The Epsilon Theorems: Simple Things Made Simple

"In the ε-calculus it is hard to understand anything"

Georg Moser

Institute of Computer Science
University of Innsbruck

Workshop on Efficient and Natural Proof Systems, Dec. 16, 2015

What is the Epsilon Calculus?

Definition

- the ε-calculus is a formalisation of logic without quantifiers but with the ε-operator

What is the Epsilon Calculus?

Definition

- the ε-calculus is a formalisation of logic without quantifiers but with the ε-operator
- if $A(x)$ is a formula, then $\varepsilon_x A(x)$ is an ε-term

What is the Epsilon Calculus?

Definition

- the ε-calculus is a formalisation of logic without quantifiers but with the ε-operator
- if $A(x)$ is a formula, then $\varepsilon_x A(x)$ is an ε-term
- $\varepsilon_x A(x)$ is an indefinite description:
  $\varepsilon_x A(x)$ is some x for which $A(x)$ is true

What is the Epsilon Calculus?

Definition

- the ε-calculus is a formalisation of logic without quantifiers but with the ε-operator
- if $A(x)$ is a formula, then $\varepsilon_x A(x)$ is an ε-term
- $\varepsilon_x A(x)$ is an indefinite description:
  $\varepsilon_x A(x)$ is some x for which $A(x)$ is true
- ε can replace $\exists$: $\exists x\, A(x) \Leftrightarrow A(\varepsilon_x A(x))$

What is the Epsilon Calculus?

Definition

- the ε-calculus is a formalisation of logic without quantifiers but with the ε-operator
- if $A(x)$ is a formula, then $\varepsilon_x A(x)$ is an ε-term
- $\varepsilon_x A(x)$ is an indefinite description:
  $\varepsilon_x A(x)$ is some x for which $A(x)$ is true
- ε can replace $\exists$: $\exists x\, A(x) \Leftrightarrow A(\varepsilon_x A(x))$
- axioms of ε-calculus:

  1. propositional tautologies
  2. identity schemata
  3. $A(t) \to A(\varepsilon_x A(x))$

What is the Epsilon Calculus?

Definition

- the ε-calculus is a formalisation of logic without quantifiers but with the ε-operator
- if $A(x)$ is a formula, then $\varepsilon_x A(x)$ is an ε-term
- $\varepsilon_x A(x)$ is an indefinite description:
  $\varepsilon_x A(x)$ is some x for which $A(x)$ is true
- ε can replace $\exists$: $\exists x\, A(x) \Leftrightarrow A(\varepsilon_x A(x))$
- axioms of ε-calculus:
  1. propositional tautologies
  2. ~~identity schemata~~ (this talk)
  3. $A(t) \rightarrow A(\varepsilon_x A(x))$

What is the Epsilon Calculus?

Definition

- the ε-calculus is a formalisation of logic without quantifiers but with the ε-operator
- if $A(x)$ is a formula, then $\varepsilon_x A(x)$ is an ε-term
- $\varepsilon_x A(x)$ is an indefinite description:
  $\varepsilon_x A(x)$ is some x for which $A(x)$ is true
- ε can replace $\exists$: $\exists x\, A(x) \Leftrightarrow A(\varepsilon_x A(x))$
- axioms of ε-calculus:
  1. propositional tautologies
  2. ~~identity schemata~~ (this talk)
  3. $A(t) \to A(\varepsilon_x A(x))$

predicate logic can be embedded in the ε-calculus

Why Should You Care?

Why Should You Care?

1. basis of proof theory

Why Should You Care?

1. basis of proof theory
2. interesting logical formalism
   - trade logical structure for term structure, that is, ε-calculus embodies deep inference ☺
   - formalisation of choice; recognised in its use in proof assistants
   - full potential TCS yet unexplored

Why Should You Care?

1. basis of proof theory
2. interesting logical formalism
   - trade logical structure for term structure, that is, ε-calculus embodies deep inference ☺
   - formalisation of choice; recognised in its use in proof assistants
   - full potential TCS yet unexplored
3. foundation of noteworthy proof-theoretic results
   - ε-theorems and Herbrand's theorem (this talk)
   - ε-substitution method and its connection to learning (Tom's talk)
   - Kreisel's no-counter example interpretation

Why Should You Care?

1. basis of proof theory

2. interesting logical formalism
   - trade logical structure for term structure, that is, ε-calculus embodies deep inference ☺
   - formalisation of choice; recognised in its use in proof assistants
   - full potential TCS yet unexplored

3. foundation of noteworthy proof-theoretic results
   - ε-theorems and Herbrand's theorem (this talk)
   - ε-substitution method and its connection to learning (Tom's talk)
   - Kreisel's no-counter example interpretation

4. you asked for it ☺:

> *I asked some of the others about the topics you proposed and there seemed to be a slight preference for epsilon calculus [...]*

Outline

- Axiomatisation
- The Embedding Lemma
- The First Epsilon Theorem
- Lower Bounds
- The Second Epsilon Theorem

Axioms of the Epsilon Calculus

Definition

- AxEC: all substitution instances of propositional tautologies
- AxEC$_\varepsilon$: AxEC + all substitution instances of

$$\underbrace{A(t) \to A(\varepsilon_x A(x))}_{\text{critical formula}}$$

- AxPC: AxEC + all substitution instances of

$$A(a) \to \exists x\, A(x) \qquad \forall x\, A(x) \to A(a)$$

- AxPC$_\varepsilon$: AxPC + all substitution instances of critical formulas

Definition

- a proof in EC (EC_ε) is a sequence $A_1, \ldots, A_n$ of formulas such that each A_i is either in AxEC ($AxEC_\varepsilon$) or it follows from formulas preceding it by modus ponens

- a proof in PC (PC_ε) is a sequence $A_1, \ldots, A_n$ of formulas such that each A_i is either in AxPC ($AxPC_\varepsilon$) or follows from formulas preceding it by modus ponens or generalisation

- if A is provable in say EC_ε we write $EC_\varepsilon \vdash_\pi A$

Definition

- a proof in EC (EC_ε) is a sequence $A_1, \dots, A_n$ of formulas such that each A_i is either in AxEC ($AxEC_\varepsilon$) or it follows from formulas preceding it by modus ponens

- a proof in PC (PC_ε) is a sequence $A_1, \dots, A_n$ of formulas such that each A_i is either in AxPC ($AxPC_\varepsilon$) or follows from formulas preceding it by modus ponens or generalisation

- if A is provable in say EC_ε we write $EC_\varepsilon \vdash_\pi A$

- the size $\mathrm{sz}(\pi)$ of a proof π is the number of steps in π

Definition

- a proof in EC (EC_ε) is a sequence $A_1, \ldots, A_n$ of formulas such that each A_i is either in AxEC ($\mathrm{AxEC}_\varepsilon$) or it follows from formulas preceding it by modus ponens
- a proof in PC (PC_ε) is a sequence $A_1, \ldots, A_n$ of formulas such that each A_i is either in AxPC ($\mathrm{AxPC}_\varepsilon$) or follows from formulas preceding it by modus ponens or generalisation
- if A is provable in say EC_ε we write $\mathrm{EC}_\varepsilon \vdash_\pi A$
- the size $\mathrm{sz}(\pi)$ of a proof π is the number of steps in π
- the critical count $\mathrm{cc}(\pi)$ of π is the number of distinct critical formulas and quantifier axioms in π (plus 1)

Definition (tentative)

quantifiers in a quantifier-free system:

$$\exists x\, A(x) \Leftrightarrow A(\varepsilon_x A(x)) \qquad \forall x\, A(x) \Leftrightarrow A(\varepsilon_x \neg A(x))$$

Definition (tentative)

quantifiers in a quantifier-free system:

$$\exists x\, A(x) \Leftrightarrow A(\varepsilon_x A(x)) \qquad \forall x\, A(x) \Leftrightarrow A(\varepsilon_x \neg A(x))$$

Definition

define a mapping $^\varepsilon$:

$$f(t_1, \dots, t_n)^\varepsilon = f(t_1^\varepsilon, \dots, t_n^\varepsilon)$$

$$x^\varepsilon = x \qquad [\varepsilon_x A(x)]^\varepsilon = \varepsilon_x A(x)^\varepsilon$$

$$a^\varepsilon = a$$

Definition (tentative)

quantifiers in a quantifier-free system:

$$\exists x\, A(x) \Leftrightarrow A(\varepsilon_x A(x)) \qquad \forall x\, A(x) \Leftrightarrow A(\varepsilon_x \neg A(x))$$

Definition

define a mapping $^{\varepsilon}$:

$$f(t_1, \ldots, t_n)^{\varepsilon} = f(t_1^{\varepsilon}, \ldots, t_n^{\varepsilon}) \qquad P(t_1, \ldots, t_n)^{\varepsilon} = P(t_1^{\varepsilon}, \ldots, t_n^{\varepsilon})$$

$$x^{\varepsilon} = x \qquad [\varepsilon_x A(x)]^{\varepsilon} = \varepsilon_x A(x)^{\varepsilon}$$

$$a^{\varepsilon} = a$$

Definition (tentative)

quantifiers in a quantifier-free system:

$$\exists x\, A(x) \Leftrightarrow A(\varepsilon_x A(x)) \qquad \forall x\, A(x) \Leftrightarrow A(\varepsilon_x \neg A(x))$$

Definition

define a mapping $^\varepsilon$:

$$f(t_1, \dots, t_n)^\varepsilon = f(t_1^\varepsilon, \dots, t_n^\varepsilon) \qquad P(t_1, \dots, t_n)^\varepsilon = P(t_1^\varepsilon, \dots, t_n^\varepsilon)$$

$$x^\varepsilon = x \qquad (A \to B)^\varepsilon = A^\varepsilon \to B^\varepsilon \qquad [\varepsilon_x A(x)]^\varepsilon = \varepsilon_x A(x)^\varepsilon$$

$$a^\varepsilon = a \qquad (A \vee B)^\varepsilon = A^\varepsilon \vee B^\varepsilon$$

$$(\neg A)^\varepsilon = \neg A^\varepsilon \qquad (A \wedge B)^\varepsilon = A^\varepsilon \wedge B^\varepsilon$$

Definition (tentative)

quantifiers in a quantifier-free system:

$$\exists x\, A(x) \Leftrightarrow A(\varepsilon_x A(x)) \qquad \forall x\, A(x) \Leftrightarrow A(\varepsilon_x \neg A(x))$$

Definition

define a mapping $^\varepsilon$:

$$f(t_1, \dots, t_n)^\varepsilon = f(t_1^\varepsilon, \dots, t_n^\varepsilon) \qquad P(t_1, \dots, t_n)^\varepsilon = P(t_1^\varepsilon, \dots, t_n^\varepsilon)$$

$$x^\varepsilon = x \qquad (A \to B)^\varepsilon = A^\varepsilon \to B^\varepsilon \qquad [\varepsilon_x A(x)]^\varepsilon = \varepsilon_x A(x)^\varepsilon$$

$$a^\varepsilon = a \qquad (A \vee B)^\varepsilon = A^\varepsilon \vee B^\varepsilon \qquad (\exists x\, A(x))^\varepsilon = A^\varepsilon(\varepsilon_x A(x)^\varepsilon)$$

$$(\neg A)^\varepsilon = \neg A^\varepsilon \qquad (A \wedge B)^\varepsilon = A^\varepsilon \wedge B^\varepsilon \qquad (\forall x\, A(x))^\varepsilon = A^\varepsilon(\varepsilon_x \neg A(x)^\varepsilon)$$

Definition (tentative)

quantifiers in a quantifier-free system:

$$\exists x\, A(x) \Leftrightarrow A(\varepsilon_x A(x)) \qquad \forall x\, A(x) \Leftrightarrow A(\varepsilon_x \neg A(x))$$

Definition

define a mapping $^\varepsilon$:

$$f(t_1,\dots,t_n)^\varepsilon = f(t_1^\varepsilon,\dots,t_n^\varepsilon) \qquad P(t_1,\dots,t_n)^\varepsilon = P(t_1^\varepsilon,\dots,t_n^\varepsilon)$$

$$x^\varepsilon = x \qquad (A \to B)^\varepsilon = A^\varepsilon \to B^\varepsilon \qquad [\varepsilon_x A(x)]^\varepsilon = \varepsilon_x A(x)^\varepsilon$$

$$a^\varepsilon = a \qquad (A \vee B)^\varepsilon = A^\varepsilon \vee B^\varepsilon \qquad (\exists x\, A(x))^\varepsilon = A^\varepsilon(\varepsilon_x A(x)^\varepsilon)$$

$$(\neg A)^\varepsilon = \neg A^\varepsilon \qquad (A \wedge B)^\varepsilon = A^\varepsilon \wedge B^\varepsilon \qquad (\forall x\, A(x))^\varepsilon = A^\varepsilon(\varepsilon_x \neg A(x)^\varepsilon)$$

Lemma

if π is a PC_ε-proof of A then there is an EC_ε-proof π^ε of A^ε with $\mathrm{sz}(\pi^\varepsilon) \leqslant 3 \cdot \mathrm{sz}(\pi)$ and $\mathrm{cc}(\pi^\varepsilon) \leqslant \mathrm{cc}(\pi)$

Example: Epsilon Mapping

Example

$$[\exists x(P(x) \quad \vee \quad \forall y\, Q(y))]^{\varepsilon} =$$

Example: Epsilon Mapping

Example

$$
\begin{aligned}
[\exists x (P(x) \quad \vee \quad & \forall y\, Q(y))]^{\varepsilon} = \\
= \quad & [P(x) \vee \forall y\, Q(y)]^{\varepsilon} \quad \{x \leftarrow \varepsilon_x [P(x) \vee \forall y\, Q(y)]^{\varepsilon}\}
\end{aligned}
$$

Example: Epsilon Mapping

Example

$$
\begin{aligned}
[\exists x(P(x) \quad \vee \quad \forall y\, Q(y))]^{\varepsilon} &= \\
&= [P(x) \vee \forall y\, Q(y)]^{\varepsilon} \quad \{x \leftarrow \varepsilon_x [P(x) \vee \forall y\, Q(y)]^{\varepsilon}\} \\
&\quad [P(x) \vee \forall y\, Q(y)]^{\varepsilon} = P(x) \vee Q(\underbrace{\varepsilon_y \neg Q(y)}_{e_1})
\end{aligned}
$$

Example: Epsilon Mapping

Example

$$[\exists x(P(x) \lor \forall y\, Q(y))]^{\varepsilon} =$$

$$= [P(x) \lor \forall y\, Q(y)]^{\varepsilon} \quad \{x \leftarrow \varepsilon_x [P(x) \lor \forall y\, Q(y)]^{\varepsilon}\}$$

$$[P(x) \lor \forall y\, Q(y)]^{\varepsilon} = P(x) \lor Q(\underbrace{\varepsilon_y \neg Q(y)}_{e_1})$$

$$= P(x) \lor Q(\underbrace{\varepsilon_y \neg Q(y)}_{e_1}) \quad \{x \leftarrow \underbrace{\varepsilon_x [P(x) \lor Q(\underbrace{\varepsilon_y \neg Q(y)}_{e_1})]}_{e_2}\}$$

Example: Epsilon Mapping

Example

$$
\begin{aligned}
[\exists x(P(x) \ \vee \ \forall y\, Q(y))]^{\varepsilon} &= \\
&= [P(x) \vee \forall y\, Q(y)]^{\varepsilon} \quad \{x \leftarrow \varepsilon_x [P(x) \vee \forall y\, Q(y)]^{\varepsilon}\} \\
&\qquad [P(x) \vee \forall y\, Q(y)]^{\varepsilon} = P(x) \vee Q(\underbrace{\varepsilon_y \neg Q(y)}_{e_1}) \\
&= P(x) \vee Q(\underbrace{\varepsilon_y \neg Q(y)}_{e_1}) \quad \{x \leftarrow \underbrace{\varepsilon_x [P(x) \vee Q(\underbrace{\varepsilon_y \neg Q(y)}_{e_1})]}_{e_2}\} \\
&= P(\underbrace{\varepsilon_x [P(x) \vee Q(\underbrace{\varepsilon_y \neg Q(y)}_{e_1})]}_{e_2}) \vee Q(\underbrace{\varepsilon_y \neg Q(y)}_{e_1})
\end{aligned}
$$

Drinker's Paradox (Yet Again)

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(a) \Rightarrow P(a)}{P(a) \Rightarrow P(a), \forall y P(y)}}{\Rightarrow P(a) \to \forall y P(y), P(a)}}{\Rightarrow \exists x (P(x) \to \forall y P(y)), P(a)}}{\Rightarrow \exists x (P(x) \to \forall y P(y)), \forall y P(y)}}{P(b) \Rightarrow \exists x (P(x) \to \forall y P(y)), \forall y P(y)}}{\Rightarrow \exists x (P(x) \to \forall y P(y)), P(b) \to \forall y P(y)}}{\Rightarrow \exists x (P(x) \to \forall y P(y)), \exists x (P(x) \to \forall y P(y))}}{\Rightarrow \exists x (P(x) \to \forall y P(y))}
$$

Drinker's Paradox (Yet Again)

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(a) \Rightarrow P(a)}{P(a) \Rightarrow P(a), \forall y P(y)}}{\Rightarrow P(a) \to \forall y P(y), P(a)}}{\Rightarrow \exists x(P(x) \to \forall y P(y)), P(a)}}{\Rightarrow \exists x(P(x) \to \forall y P(y)), \forall y P(y)}}{P(b) \Rightarrow \exists x(P(x) \to \forall y P(y)), \forall y P(y)}}{\Rightarrow \exists x(P(x) \to \forall y P(y)), P(b) \to \forall y P(y)}}{\Rightarrow \exists x(P(x) \to \forall y P(y)), \exists x(P(x) \to \forall y P(y))}}{\Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y))}$$

where we employ

$$[\forall y P(y)]^\varepsilon = P(\varepsilon_y \neg P(y))$$

$$[\exists x(P(x) \to \forall y P(y)]^\varepsilon = P(\underbrace{\varepsilon_x(P(x) \to P(\varepsilon_y \neg P(y)))}_{\varepsilon}) \to P(\varepsilon_y \neg P(y))$$

Drinker's Paradox (Yet Again)

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(a) \Rightarrow P(a)}{P(a) \Rightarrow P(a), \forall y P(y)}}{\Rightarrow P(a) \to \forall y P(y), P(a)}}{\Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y)), P(a)}}{\Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y)), \forall y P(y)}}{P(b) \Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y)), \forall y P(y)}}{\Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y)), P(b) \to \forall y P(y)}}{\Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y)), P(\varepsilon) \to P(\varepsilon_y \neg P(y))}}{\Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y))}$$

where we employ

$$[\forall y P(y)]^{\varepsilon} = P(\varepsilon_y \neg P(y))$$

$$[\exists x (P(x) \to \forall y P(y)]^{\varepsilon} = P(\underbrace{\varepsilon_x (P(x) \to P(\varepsilon_y \neg P(y)))}_{\varepsilon}) \to P(\varepsilon_y \neg P(y))$$

Drinker's Paradox (Yet Again)

Example

$$
\begin{array}{c}
P(a) \Rightarrow P(a) \\
\hline
P(a) \Rightarrow P(a), P(\varepsilon_y \neg P(y)) \\
\hline
\Rightarrow P(a) \rightarrow P(\varepsilon_y \neg P(y)), P(a) \\
\hline
\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(a) \\
\hline
\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y)) \\
\hline
P(b) \Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y)) \\
\hline
\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(b) \rightarrow P(\varepsilon_y \neg P(y)) \\
\hline
\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)) \\
\hline
\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y))
\end{array}
$$

where we employ

$$[\forall y P(y)]^\varepsilon = P(\varepsilon_y \neg P(y))$$

$$[\exists x (P(x) \rightarrow \forall y P(y)]^\varepsilon = P(\underbrace{\varepsilon_x (P(x) \rightarrow P(\varepsilon_y \neg P(y)))}_{\varepsilon}) \rightarrow P(\varepsilon_y \neg P(y))$$

Drinker's Paradox (Yet Again)

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(\varepsilon_y \neg P(y)) \Rightarrow P(\varepsilon_y \neg P(y))}{P(\varepsilon_y \neg P(y)) \Rightarrow P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y))}}{\Rightarrow P(\varepsilon_y \neg P(y)) \rightarrow P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y))}}{\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y))}}{\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y))}}{P(\varepsilon) \Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y))}}{\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y))}}{\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y))}}{\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y))}$$

where we employ

$$[\forall y P(y)]^\varepsilon = P(\varepsilon_y \neg P(y))$$

$$[\exists x (P(x) \rightarrow \forall y P(y)]^\varepsilon = P(\underbrace{\varepsilon_x (P(x) \rightarrow P(\varepsilon_y \neg P(y)))}_{\varepsilon}) \rightarrow P(\varepsilon_y \neg P(y))$$

Drinker's Paradox (Yet Again)

Example

$$\begin{array}{c}
P(\varepsilon_y \neg P(y)) \Rightarrow P(\varepsilon_y \neg P(y)) \\ \hline
P(\varepsilon_y \neg P(y)) \Rightarrow P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y)) \\ \hline
\Rightarrow P(\varepsilon_y \neg P(y)) \to P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y)) \\ \hline
\Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y)) \\ \hline
\Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y)) \\ \hline
P(\varepsilon) \Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y)) \\ \hline
\Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y)), P(\varepsilon) \to P(\varepsilon_y \neg P(y)) \\ \hline
\Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y)), P(\varepsilon) \to P(\varepsilon_y \neg P(y)) \\ \hline
\Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y))
\end{array}$$

where we employ

$$[\forall y P(y)]^\varepsilon = P(\varepsilon_y \neg P(y))$$

$$[\exists x (P(x) \to \forall y P(y)]^\varepsilon = P(\underbrace{\varepsilon_x (P(x) \to P(\varepsilon_y \neg P(y)))}_{\varepsilon}) \to P(\varepsilon_y \neg P(y))$$

Drinker's Paradox (Yet Again)

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(\varepsilon_y \neg P(y)) \Rightarrow P(\varepsilon_y \neg P(y))}{P(\varepsilon_y \neg P(y)) \Rightarrow P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y))}}{\Rightarrow P(\varepsilon_y \neg P(y)) \to P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y))}}{\Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y))}}{\Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y))}}{P(\varepsilon) \Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y))}}{\Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y)), P(\varepsilon) \to P(\varepsilon_y \neg P(y))}}{\Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y)), P(\varepsilon) \to P(\varepsilon_y \neg P(y))}}{\Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y))}$$

where we employ

$$[P(\varepsilon_y \neg P(y)) \to P(\varepsilon_y \neg P(y))] \to [P(\underbrace{\varepsilon_x(P(x) \to P(\varepsilon_y \neg P(y)))}_{\varepsilon}) \to P(\varepsilon_y \neg P(y))]$$

Drinker's Paradox (à la Michel Parigot)

Example (cont'd)

1	$P(\varepsilon_y \neg P(y)) \to P(\varepsilon_y \neg P(y))$	TAUT
2	$(P(\varepsilon_y \neg P(y)) \to P(\varepsilon_y \neg P(y))) \to$	
	$\to (P(\varepsilon_x (P(x) \to P(\varepsilon_y \neg P(y))))) \to P(\varepsilon_y \neg P(y)))$	critical axiom
3	$P(\varepsilon_x (P(x) \to P(\varepsilon_y \neg P(y))))) \to P(\varepsilon_y \neg P(y))$	$1, 2, MP$

Drinker's Paradox (à la Michel Parigot)

Example (cont'd)

1	$P(\varepsilon_y \neg P(y)) \to P(\varepsilon_y \neg P(y))$	TAUT
2	$(P(\varepsilon_y \neg P(y)) \to P(\varepsilon_y \neg P(y))) \to$ $\to (P(\varepsilon_x(P(x) \to P(\varepsilon_y \neg P(y))))) \to P(\varepsilon_y \neg P(y)))$	critical axiom
3	$P(\varepsilon_x(P(x) \to P(\varepsilon_y \neg P(y))))) \to P(\varepsilon_y \neg P(y))$	$1, 2, MP$

Example (recall Michel's talk)

$$\dfrac{\dfrac{\Rightarrow P(a) \to P(a)}{\Rightarrow P(v) \to \forall y P(y)}}{\Rightarrow \exists x(P(x) \to \forall y P(y))}$$

Drinker's Paradox (à la Michel Parigot)

Example (cont'd)

1	$P(\varepsilon_y \neg P(y)) \to P(\varepsilon_y \neg P(y))$	TAUT
2	$(P(\varepsilon_y \neg P(y)) \to P(\varepsilon_y \neg P(y))) \to$ $\to (P(\varepsilon_x(P(x) \to P(\varepsilon_y \neg P(y))))) \to P(\varepsilon_y \neg P(y)))$	critical axiom
3	$P(\varepsilon_x(P(x) \to P(\varepsilon_y \neg P(y))))) \to P(\varepsilon_y \neg P(y))$	$1, 2, MP$

Example (recall Michel's talk)

$$\cfrac{\cfrac{\Rightarrow P(\varepsilon_y \neg P(y)) \to P(\varepsilon_y \neg P(y))}{\Rightarrow P(\varepsilon_y \neg P(y)) \to \forall y P(y)}}{\Rightarrow \exists x (P(x) \to \forall y P(y))}$$

Proof of the Lemma

Proof

- we show $\forall$ proofs $\pi\colon A_1, \ldots, A_n$
  $\exists$ proof π^ε containing $A_1^\varepsilon, \ldots, A_n^\varepsilon$ (+ extra formulas)
- we use by induction on n

Proof of the Lemma

Proof

- we show $\forall$ proofs $\pi\colon A_1, \dots, A_n$
  $\exists$ proof π^ε containing $A_1^\varepsilon, \dots, A_n^\varepsilon$ (+ extra formulas)
- we use by induction on n
- base case is trivial and if $A_n =: A$ is a propositional tautology, A^ε is also a tautology

Proof of the Lemma

Proof

- we show $\forall$ proofs π: $A_1, \dots, A_n$
  $\exists$ proof π^ε containing $A_1^\varepsilon, \dots, A_n^\varepsilon$ (+ extra formulas)
- we use by induction on n
- base case is trivial and if $A_n =: A$ is a propositional tautology, A^ε is also a tautology
- Case A an instance of a quantifier axiom; suppose
  $A = A(t) \to \exists x A(x)$; hence

$$[A(t) \to \exists x\, A(x)]^\varepsilon = A^\varepsilon(t^\varepsilon) \to A^\varepsilon(\varepsilon_x A(x)^\varepsilon)$$

  the latter is a critical axiom

Proof of the Lemma

Proof

- we show $\forall$ proofs π: $A_1, \dots, A_n$
  $\exists$ proof π^ε containing $A_1^\varepsilon, \dots, A_n^\varepsilon$ (+ extra formulas)
- we use by induction on n
- base case is trivial and if $A_n =: A$ is a propositional tautology, A^ε is also a tautology
- Case A an instance of a quantifier axiom; suppose
  $A = A(t) \to \exists x A(x)$; hence

$$[A(t) \to \exists x\, A(x)]^\varepsilon = A^\varepsilon(t^\varepsilon) \to A^\varepsilon(\varepsilon_x A(x)^\varepsilon)$$

  the latter is a critical axiom
- Case A follows by modus ponens from A_i and $A_j \equiv A_i \to A$
  applying IH there exists a proof π^* containing A_i^ε and $A_i^\varepsilon \to A_j^\varepsilon$; we add A^ε to π^*

Proof (cont'd).

- Case A follows by generalisation; i.e. $A = B \to \forall x\, C(x)$ and there exists $A_i = B \to C(a)$; a eigenvariable
  by IH there exists a proof π^* containing $A_i^\varepsilon \equiv B^\varepsilon \to C(a)^\varepsilon$; replacing the eigenvariable a by $\varepsilon_x \neg A^\varepsilon(x)$ results in a proof containing

$$B^\varepsilon \to A^\varepsilon(\varepsilon_x \neg A^\varepsilon(x)) = [B \to \forall x\, C(x)]^\varepsilon$$

  we set $\pi^\varepsilon := \pi^*\{a \mapsto \varepsilon_x \neg A^\varepsilon(x)\}$

Proof (cont'd).

- Case A follows by generalisation; i.e. $A = B \to \forall x\, C(x)$ and there exists $A_i = B \to C(a)$; a eigenvariable
  by IH there exists a proof π^* containing $A_i^\varepsilon \equiv B^\varepsilon \to C(a)^\varepsilon$; replacing the eigenvariable a by $\varepsilon_x \neg A^\varepsilon(x)$ results in a proof containing

$$B^\varepsilon \to A^\varepsilon(\varepsilon_x \neg A^\varepsilon(x)) = [B \to \forall x\, C(x)]^\varepsilon$$

  we set $\pi^\varepsilon := \pi^*\{a \mapsto \varepsilon_x \neg A^\varepsilon(x)\}$

Proof (cont'd).

- Case A follows by generalisation; i.e. $A = B \to \forall x\, C(x)$ and there exists $A_i = B \to C(a)$; a eigenvariable
  by IH there exists a proof π^* containing $A_i^\varepsilon \equiv B^\varepsilon \to C(a)^\varepsilon$; replacing the eigenvariable a by $\varepsilon_x \neg A^\varepsilon(x)$ results in a proof containing

$$B^\varepsilon \to A^\varepsilon(\varepsilon_x \neg A^\varepsilon(x)) = [B \to \forall x\, C(x)]^\varepsilon$$

  we set $\pi^\varepsilon := \pi^*\{a \mapsto \varepsilon_x \neg A^\varepsilon(x)\}$

■

Lemma (Embedding Lemma)

if π is a PC_ε-proof of A then there is an EC_ε-proof π^ε of A^ε with $\mathrm{sz}(\pi^\varepsilon) \leqslant 3 \cdot \mathrm{sz}(\pi)$ *and* $\mathrm{cc}(\pi^\varepsilon) \leqslant \mathrm{cc}(\pi)$

Proof (cont'd).

- Case A follows by generalisation; i.e. $A = B \to \forall x\, C(x)$ and there exists $A_i = B \to C(a)$; a eigenvariable
  by IH there exists a proof π^* containing $A_i^\varepsilon \equiv B^\varepsilon \to C(a)^\varepsilon$; replacing the eigenvariable a by $\varepsilon_x \neg A^\varepsilon(x)$ results in a proof containing

$$B^\varepsilon \to A^\varepsilon(\varepsilon_x \neg A^\varepsilon(x)) = [B \to \forall x\, C(x)]^\varepsilon$$

  we set $\pi^\varepsilon := \pi^*\{a \mapsto \varepsilon_x \neg A^\varepsilon(x)\}$

Lemma (Embedding Lemma)

if π is a PC_ε-proof of A then there is an EC_ε-proof π^ε of A^ε with $\mathrm{sz}(\pi^\varepsilon) \leqslant 3 \cdot \mathrm{sz}(\pi)$ *and* $\mathrm{cc}(\pi^\varepsilon) \leqslant \mathrm{cc}(\pi)$

Quiz

Question

the proof of the embedding lemma is wrong; can you spot the mistake?

Quiz

Question

the proof of the embedding lemma is wrong; can you spot the mistake?

Answer

the application of IH in the generalisation case requires more work[a]

[a] paper by M., Zach contains the presented proof; bug was spotted by Michel Parigot, thank!

The First Epsilon Theorem

Theorem

suppose $E(e_1, \ldots, e_m)$ is a quantifier-free formula containing only the ε-terms s_1, ..., s_m, and

$$\mathrm{EC}_\varepsilon \vdash_\pi E(s_1, \ldots, s_m) \ ,$$

then there are ε-free terms t_j^i such that

$$\mathrm{EC} \vdash \bigvee_{i=1}^{n} E(t_1^i, \ldots, t_m^i)$$

The First Epsilon Theorem

Theorem

suppose $E(e_1, \ldots, e_m)$ is a quantifier-free formula containing only the ε-terms $s_1, \ldots, s_m$, and

$$\mathsf{EC}_\varepsilon \vdash_\pi E(s_1, \ldots, s_m) \ ,$$

then there are ε-free terms t^i_j such that

$$\mathsf{EC} \vdash \bigvee_{i=1}^{n} E(t^i_1, \ldots, t^i_m)$$

where $n \leqslant 2^{3 \cdot \mathrm{cc}(\pi)}_{2 \cdot \mathrm{cc}(\pi)}$

The First Epsilon Theorem

Theorem

suppose $E(e_1, \ldots, e_m)$ is a quantifier-free formula containing only the ε-terms s_1, ..., s_m, and

$$\mathrm{EC}_\varepsilon \vdash_\pi E(s_1, \ldots, s_m) \ ,$$

then there are ε-free terms t^i_j such that

$$\mathrm{EC} \vdash \bigvee_{i=1}^{n} E(t^i_1, \ldots, t^i_m)$$

where $n \leqslant 2^{3\cdot \mathrm{cc}(\pi)}_{2\cdot \mathrm{cc}(\pi)}$

number of instances independent off # of propositional inferences

Herbrand's Theorem

Theorem

if $\exists x_1 \dots \exists x_m E(x_1, \dots, x_m)$ *is a purely existential formula containing only the bound variables* x_1, ..., x_m, *and*

$$\mathrm{PC} \vdash_\pi \exists x_1 \dots \exists x_m E(x_1, \dots, x_m) ,$$

then there are *terms* t^i_j *such that*

$$\mathrm{EC} \vdash \bigvee_{i=1}^{n} E(t^i_1, \dots, t^i_m)$$

Herbrand's Theorem

Theorem

if $\exists x_1 \dots \exists x_m E(x_1, \dots, x_m)$ *is a purely existential formula containing only the bound variables* $x_1, \dots, x_m$*, and*

$$\mathrm{PC} \vdash_\pi \exists x_1 \dots \exists x_m E(x_1, \dots, x_m) ,$$

then there are *terms* t^i_j *such that*

$$\mathrm{EC} \vdash \bigvee_{i=1}^{n} E(t^i_1, \dots, t^i_m)$$

where $n \leqslant 2^{3 \cdot \mathrm{cc}(\pi)}_{2 \cdot \mathrm{cc}(\pi)}$

Herbrand's Theorem

Theorem

if $\exists x_1 \ldots \exists x_m E(x_1, \ldots, x_m)$ *is a purely existential formula containing only the bound variables* $x_1, \ldots, x_m$*, and*

$$\mathrm{PC} \vdash_\pi \exists x_1 \ldots \exists x_m E(x_1, \ldots, x_m) \ ,$$

then there are *terms* t^i_j *such that*

$$\mathrm{EC} \vdash \bigvee_{i=1}^{n} E(t^i_1, \ldots, t^i_m)$$

where $n \leqslant 2^{3 \cdot \mathrm{cc}(\pi)}_{2 \cdot \mathrm{cc}(\pi)}$

length of Herbrand disjunction independent off # of propositional inferences

Herbrand's Theorem

Theorem

if $\exists x_1 \dots \exists x_m E(x_1, \dots, x_m)$ *is a purely existential formula containing only the bound variables* $x_1, \dots, x_m$*, and*

$$\mathrm{PC}_\varepsilon \vdash_\pi \exists x_1 \dots \exists x_m E(x_1, \dots, x_m) \ ,$$

then there are ε*-free terms* t^i_j *such that*

$$\mathrm{EC} \vdash \bigvee_{i=1}^{n} E(t^i_1, \dots, t^i_m)$$

where $n \leqslant 2^{3 \cdot \mathrm{cc}(\pi)}_{2 \cdot \mathrm{cc}(\pi)}$

length of Herbrand disjunction independent off # of propositional inferences

Lower Bounds

Observations

- the upper bound on the length of the Herbrand disjunction depends only on the critical count of the initial proof
- in contrast, usually the bound depends on the length and cut complexity of the original proof

Lower Bounds

Observations

- the upper bound on the length of the Herbrand disjunction depends only on the critical count of the initial proof
- in contrast, usually the bound depends on the length and cut complexity of the original proof
- in both cases the relationship is hyperexponential
- its well-known that proofs with cut have hyper-exponential speedup over cut-free proofs

Lower Bounds

Observations

- the upper bound on the length of the Herbrand disjunction depends only on the critical count of the initial proof
- in contrast, usually the bound depends on the length and cut complexity of the original proof
- in both cases the relationship is hyperexponential
- its well-known that proofs with cut have hyper-exponential speedup over cut-free proofs

Question

what about lower-bounds of the ε-elimination procedure

Lower Bounds

Definition

- an $\vee$-expansion (of $E \equiv E(s_1, \ldots, s_m)$) is a finite disjunction

$$E' \equiv E_1 \vee \cdots \vee E_l$$

$E_i \equiv E(s_1^i, \ldots, s_m^i)$ for terms s_j^i

Lower Bounds

Definition

- an $\vee$-expansion (of $E \equiv E(s_1, \ldots, s_m)$) is a finite disjunction
$$E' \equiv E_1 \vee \cdots \vee E_l$$
$E_i \equiv E(s^i_1, \ldots, s^i_m)$ for terms s^i_j
- the Herbrand complexity $\mathrm{HC}(E)$ of a purely existential formula $E \equiv \exists x_1 \ldots \exists x_n E'(x_1, \ldots, x_n)$ is the length of the shortest valid $\vee$-expansion of $E'(x_1, \ldots, x_n)$

Lower Bounds

Definition

- an $\vee$-expansion (of $E \equiv E(s_1, \ldots, s_m)$) is a finite disjunction
$$E' \equiv E_1 \vee \cdots \vee E_l$$
$E_i \equiv E(s^i_1, \ldots, s^i_m)$ for terms s^i_j
- the Herbrand complexity $\mathrm{HC}(E)$ of a purely existential formula $E \equiv \exists x_1 \ldots \exists x_n E'(x_1, \ldots, x_n)$ is the length of the shortest valid $\vee$-expansion of $E'(x_1, \ldots, x_n)$

Theorem

there is a sequence of formulas E_k so that

1. *for each k, $\exists$ PC_ε-proof π_k of E_k with $\mathrm{cc}(\pi_k) \leq c \cdot k$, but*
2. $\mathrm{HC}(E_k) \geq 2^1_k$.

Proof Sketch of Part 1

Definition

$$\mathrm{Hyp}_1 := \forall x\, R(x, 0, S(x))$$

$$\mathrm{Hyp}_2 := \forall y \forall x \forall z \forall z_1 (R(y, x, z) \wedge R(z, x, z_1) \rightarrow R(y, S(x), z_1))$$

$$C_k := \exists z_k \dots \exists z_0 (R(0, 0, z_k) \wedge R(0, z_k, z_{k-1}) \wedge \cdots \wedge R(0, z_1, z_0))$$

E_k := (purely existential) prefix form of $\mathrm{Hyp}_1 \wedge \mathrm{Hyp}_2 \rightarrow C_k$

Proof Sketch of Part 1

Definition

$$\mathrm{Hyp}_1 := \forall x\, R(x, 0, S(x))$$

$$\mathrm{Hyp}_2 := \forall y \forall x \forall z \forall z_1 (R(y, x, z) \wedge R(z, x, z_1) \to R(y, S(x), z_1))$$

$$C_k := \exists z_k \dots \exists z_0 (R(0, 0, z_k) \wedge R(0, z_k, z_{k-1}) \wedge \cdots \wedge R(0, z_1, z_0))$$

$$E_k := \text{(purely existential) prefix form of } \mathrm{Hyp}_1 \wedge \mathrm{Hyp}_2 \to C_k$$

$R(n, m, k)$ expresses that $n + 2^m = k$, and C_k expresses that 2^1_k is defined

Proof Sketch of Part 1

Definition

$$\mathrm{Hyp}_1 := \forall x\, R(x, 0, S(x))$$

$$\mathrm{Hyp}_2 := \forall y \forall x \forall z \forall z_1 (R(y, x, z) \wedge R(z, x, z_1) \to R(y, S(x), z_1))$$

$$C_k := \exists z_k \dots \exists z_0 (R(0, 0, z_k) \wedge R(0, z_k, z_{k-1}) \wedge \cdots \wedge R(0, z_1, z_0))$$

$$E_k := \text{(purely existential) prefix form of } \mathrm{Hyp}_1 \wedge \mathrm{Hyp}_2 \to C_k$$

$R(n, m, k)$ expresses that $n + 2^m = k$, and C_k expresses that 2^1_k is defined

Lemma

for every k, $\mathsf{PC}_\varepsilon \vdash_{\pi_k} E_k$, where $\mathrm{cc}(\pi_k) = c \cdot k$

Proof Sketch of Part 1

Definition

$$\mathrm{Hyp}_1 := \forall x\, R(x, 0, S(x))$$

$$\mathrm{Hyp}_2 := \forall y \forall x \forall z \forall z_1 (R(y, x, z) \land R(z, x, z_1) \to R(y, S(x), z_1))$$

$$C_k := \exists z_k \dots \exists z_0 (R(0, 0, z_k) \land R(0, z_k, z_{k-1}) \land \cdots \land R(0, z_1, z_0))$$

$$E_k := \text{(purely existential) prefix form of } \mathrm{Hyp}_1 \land \mathrm{Hyp}_2 \to C_k$$

$R(n, m, k)$ expresses that $n + 2^m = k$, and C_k expresses that 2^1_k is defined

Lemma

for every k, $\mathrm{PC}_\varepsilon \vdash_{\pi_k} E_k$, *where* $\mathrm{cc}(\pi_k) = c \cdot k$

this establishes part one of the theorem

Proof Sketch of Part 2

Definition

consider Herbrand sequents of the sequent $\mathrm{Hyp}_1, \mathrm{Hyp}_2 \Rightarrow C_k$

1. each of these sequents has the from $\Gamma_1, \Gamma_2 \Rightarrow \Delta$

Proof Sketch of Part 2

Definition

consider Herbrand sequents of the sequent $\mathrm{Hyp}_1, \mathrm{Hyp}_2 \Rightarrow C_k$

1. each of these sequents has the from $\Gamma_1, \Gamma_2 \Rightarrow \Delta$ such that each formula in
   - Γ_1 is instance of $R(x, 0, S(x))$
   - Γ_2 is instance of $R(y, x, z) \wedge R(z, x, z_1) \rightarrow R(y, S(x), z_1)$
   - Δ is instance of $R(0, 0, z_k) \wedge R(0, z_k, z_{k-1}) \wedge \cdots \wedge R(0, z_1, z_0)$

Proof Sketch of Part 2

Definition

consider Herbrand sequents of the sequent $\mathrm{Hyp}_1, \mathrm{Hyp}_2 \Rightarrow C_k$

1. each of these sequents has the from $\Gamma_1, \Gamma_2 \Rightarrow \Delta$ such that each formula in
   - Γ_1 is instance of $R(x, 0, S(x))$
   - Γ_2 is instance of $R(y, x, z) \wedge R(z, x, z_1) \to R(y, S(x), z_1)$
   - Δ is instance of $R(0, 0, z_k) \wedge R(0, z_k, z_{k-1}) \wedge \cdots \wedge R(0, z_1, z_0)$
2. $\max\{|\Gamma_1|, |\Gamma_2|, |\Delta|\} \leqslant \mathrm{HC}(E_k)$

Proof Sketch of Part 2

Definition

consider Herbrand sequents of the sequent $\mathrm{Hyp}_1, \mathrm{Hyp}_2 \Rightarrow C_k$

1. each of these sequents has the from $\Gamma_1, \Gamma_2 \Rightarrow \Delta$ such that each formula in
   - Γ_1 is instance of $R(x, 0, S(x))$
   - Γ_2 is instance of $R(y, x, z) \wedge R(z, x, z_1) \rightarrow R(y, S(x), z_1)$
   - Δ is instance of $R(0, 0, z_k) \wedge R(0, z_k, z_{k-1}) \wedge \cdots \wedge R(0, z_1, z_0)$
2. $\max\{|\Gamma_1|, |\Gamma_2|, |\Delta|\} \leqslant \mathrm{HC}(E_k)$

Lemma

if $T = (\Gamma_1, \Gamma_2 \Rightarrow \Delta)$ *is a minimal Herbrand sequent of* $\mathrm{Hyp}_1, \mathrm{Hyp}_2 \Rightarrow C_k$*, then* $|\Gamma_1| \geqslant 2^1_k$

Proof Sketch of Part 2

Definition

consider Herbrand sequents of the sequent $\mathrm{Hyp}_1, \mathrm{Hyp}_2 \Rightarrow C_k$

1. each of these sequents has the from $\Gamma_1, \Gamma_2 \Rightarrow \Delta$ such that each formula in
   - Γ_1 is instance of $R(x, 0, S(x))$
   - Γ_2 is instance of $R(y, x, z) \wedge R(z, x, z_1) \to R(y, S(x), z_1)$
   - Δ is instance of $R(0, 0, z_k) \wedge R(0, z_k, z_{k-1}) \wedge \cdots \wedge R(0, z_1, z_0)$
2. $\max\{|\Gamma_1|, |\Gamma_2|, |\Delta|\} \leqslant \mathrm{HC}(E_k)$

Lemma

if $T = (\Gamma_1, \Gamma_2 \Rightarrow \Delta)$ *is a minimal Herbrand sequent of* $\mathrm{Hyp}_1, \mathrm{Hyp}_2 \Rightarrow C_k$*, then* $|\Gamma_1| \geqslant 2^1_k$

this establishes the lower bound on $\mathrm{HC}(E_k)$

The Second Epsilon Theorem

Theorem

If A is a formula of $L(\mathrm{PC})$ and $\mathrm{PC}_\varepsilon \vdash A$, then $\mathrm{PC} \vdash A$.

The Second Epsilon Theorem

Theorem

If A is a formula of $L(\mathrm{PC})$ and $\mathrm{PC}_\varepsilon \vdash A$, then $\mathrm{PC} \vdash A$.

Proof Sketch

- assume $A = \exists x \forall y \exists z\, B(x, y, z)$ and $\mathrm{PC}_\varepsilon \vdash A$

The Second Epsilon Theorem

Theorem

If A is a formula of $L(\mathrm{PC})$ and $\mathrm{PC}_\varepsilon \vdash A$, then $\mathrm{PC} \vdash A$.

Proof Sketch

- assume $A = \exists x \forall y \exists z\, B(x, y, z)$ and $\mathrm{PC}_\varepsilon \vdash A$
- then $\mathrm{PC}_\varepsilon \vdash \exists x \exists z\, B(x, f(x), z) =: A^H$

The Second Epsilon Theorem

Theorem

If A is a formula of $L(\mathrm{PC})$ and $\mathrm{PC}_\varepsilon \vdash A$, then $\mathrm{PC} \vdash A$.

Proof Sketch

- assume $A = \exists x \forall y \exists z\, B(x, y, z)$ and $\mathrm{PC}_\varepsilon \vdash A$
- then $\mathrm{PC}_\varepsilon \vdash \exists x \exists z\, B(x, f(x), z) =: A^H$
- apply the first epsilon theorem to this formula: $\exists$ ε-free terms r_i, s_i

$$\mathrm{EC} \vdash \bigvee_i B(r_i, f(r_i), s_i)$$

The Second Epsilon Theorem

Theorem

If A is a formula of $L(\mathrm{PC})$ and $\mathrm{PC}_\varepsilon \vdash A$, then $\mathrm{PC} \vdash A$.

Proof Sketch

- assume $A = \exists x \forall y \exists z\, B(x, y, z)$ and $\mathrm{PC}_\varepsilon \vdash A$
- then $\mathrm{PC}_\varepsilon \vdash \exists x \exists z\, B(x, f(x), z) =: A^H$
- apply the first epsilon theorem to this formula: $\exists$ ε-free terms r_i, s_i

$$\mathrm{EC} \vdash \bigvee_i B(r_i, f(r_i), s_i)$$

- replace the $f(r_i)$ by fresh free variables a_i such that $\mathrm{EC} \vdash \bigvee_i B(r_i', a_i, s_i')$

The Second Epsilon Theorem

Theorem

If A is a formula of $L(\mathrm{PC})$ and $\mathrm{PC}_\varepsilon \vdash A$, then $\mathrm{PC} \vdash A$.

Proof Sketch

- assume $A = \exists x \forall y \exists z\, B(x, y, z)$ and $\mathrm{PC}_\varepsilon \vdash A$
- then $\mathrm{PC}_\varepsilon \vdash \exists x \exists z\, B(x, f(x), z) =: A^H$
- apply the first epsilon theorem to this formula: $\exists$ ε-free terms r_i, s_i

$$\mathrm{EC} \vdash \bigvee_i B(r_i, f(r_i), s_i)$$

- replace the $f(r_i)$ by fresh free variables a_i such that $\mathrm{EC} \vdash \bigvee_i B(r_i', a_i, s_i')$
- deduce A in PC from above disjunction, essentially applying quantifier shiftings

The Second Epsilon Theorem

Theorem

If A is a formula of $L(\mathrm{PC})$ and $\mathrm{PC}_\varepsilon \vdash A$, then $\mathrm{PC} \vdash A$.

Proof Sketch

- assume $A = \exists x \forall y \exists z\, B(x,y,z)$ and $\mathrm{PC}_\varepsilon \vdash A$
- then $\mathrm{PC}_\varepsilon \vdash \exists x \exists z\, B(x, f(x), z) =: A^H$
- apply the first epsilon theorem to this formula: $\exists$ ε-free terms r_i, s_i

$$\mathrm{EC} \vdash \bigvee_i B(r_i, f(r_i), s_i)$$

- replace the $f(r_i)$ by fresh free variables a_i such that $\mathrm{EC} \vdash \bigvee_i B(r_i', a_i, s_i')$
- deduce A in PC from above disjunction, essentially applying quantifier shiftings

Conclusion and Future Work

Final Remarks

1. we only treated the case without equality

2. ε-theorems and Herbrand's theorem: proof theory without sequents
   - the bound on the length of the Herbrand disjunction depends only on the critical count of the initial proof

Conclusion and Future Work

Final Remarks

1. we only treated the case without equality
2. ε-theorems and Herbrand's theorem: proof theory without sequents
   - the bound on the length of the Herbrand disjunction depends only on the critical count of the initial proof

Future Work

finally sort out the case with equality:

1. equality is represented by (ε-)identity schema
2. known method for ε-elimination approximates possible size of atom formulas
3. destroys exclusive dependency of length of Herbrand disjunction on critical count

A Big Thank You to Alessio, Anupam, Lutz, Paola, and Willem for this Exciting Workshop!

... and Thanks All of You for Your Attention!