

The Epsilon Theorems: Simple Things Made Simple

“In the ε -calculus it is hard to understand anything”

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What is the Epsilon Calculus?

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 - 1 propositional tautologies
 - 2 identity schemata
 - 3 $A(t) \rightarrow A(\varepsilon_x A(x))$

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predicate logic can be embedded in the ε -calculus

Why Should You Care?



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1 basis of proof theory



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2 interesting logical formalism

- trade logical structure for term structure, that is, ε -calculus embodies deep inference ☺
- formalisation of choice; recognised in its use in proof assistants
- full potential TCS yet unexplored



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3 foundation of noteworthy proof-theoretic results

- ε -theorems and Herbrand's theorem (this talk)
- ε -substitution method and its connection to learning (Tom's talk)
- Kreisel's no-counter example interpretation

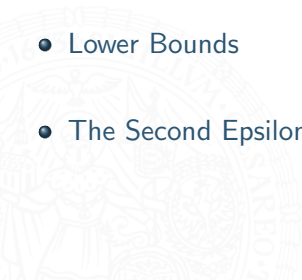
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- 4 you asked for it ☺:

I asked some of the others about the topics you proposed and there seemed to be a slight preference for epsilon calculus [...]

Outline

- Axiomatisation
- The Embedding Lemma
- The First Epsilon Theorem
- Lower Bounds
- The Second Epsilon Theorem



Axioms of the Epsilon Calculus

Definition

- **AxEC**: all substitution instances of propositional tautologies
- **AxEC_ε**: **AxEC** + all substitution instances of

$$\underbrace{A(t) \rightarrow A(\varepsilon_x A(x))}_{\text{critical formula}}$$

- **AxPC**: **AxEC** + all substitution instances of

$$A(a) \rightarrow \exists x A(x) \quad \forall x A(x) \rightarrow A(a)$$

- **AxPC_ε**: **AxPC** + all substitution instances of critical formulas

Definition

- a **proof** in **EC** (EC_ε) is a sequence A_1, \dots, A_n of formulas such that each A_i is either in $AxEC$ ($AxEC_\varepsilon$) or it follows from formulas preceding it by **modus ponens**
- a **proof** in **PC** (PC_ε) is a sequence A_1, \dots, A_n of formulas such that each A_i is either in $AxPC$ ($AxPC_\varepsilon$) or follows from formulas preceding it by **modus ponens** or **generalisation**
- if A is provable in say EC_ε we write $EC_\varepsilon \vdash_\pi A$

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- if A is provable in say EC_ε we write $EC_\varepsilon \vdash_\pi A$
- the size $SZ(\pi)$ of a proof π is the number of steps in π
- the **critical count** $cc(\pi)$ of π is the number of distinct critical formulas and quantifier axioms in π (plus 1)

Definition (tentative)

quantifiers in a quantifier-free system:

$$\exists x A(x) \Leftrightarrow A(\varepsilon_x A(x)) \quad \forall x A(x) \Leftrightarrow A(\varepsilon_x \neg A(x))$$



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Definition

define a **mapping** ε :

$$f(t_1, \dots, t_n)^\varepsilon = f(t_1^\varepsilon, \dots, t_n^\varepsilon)$$

$$x^\varepsilon = x$$

$$a^\varepsilon = a$$

$$[\varepsilon_x A(x)]^\varepsilon = \varepsilon_x A(x)^\varepsilon$$

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$$x^\varepsilon = x \quad (A \rightarrow B)^\varepsilon = A^\varepsilon \rightarrow B^\varepsilon \quad [\varepsilon_x A(x)]^\varepsilon = \varepsilon_x A(x)^\varepsilon$$

$$a^\varepsilon = a \quad (A \vee B)^\varepsilon = A^\varepsilon \vee B^\varepsilon$$

$$(\neg A)^\varepsilon = \neg A^\varepsilon \quad (A \wedge B)^\varepsilon = A^\varepsilon \wedge B^\varepsilon$$

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 x^\varepsilon = x \quad (A \rightarrow B)^\varepsilon = A^\varepsilon \rightarrow B^\varepsilon \quad [\varepsilon_x A(x)]^\varepsilon = \varepsilon_x A(x)^\varepsilon \\
 a^\varepsilon = a \quad (A \vee B)^\varepsilon = A^\varepsilon \vee B^\varepsilon \quad (\exists x A(x))^\varepsilon = A^\varepsilon(\varepsilon_x A(x)^\varepsilon) \\
 (\neg A)^\varepsilon = \neg A^\varepsilon \quad (A \wedge B)^\varepsilon = A^\varepsilon \wedge B^\varepsilon \quad (\forall x A(x))^\varepsilon = A^\varepsilon(\varepsilon_x \neg A(x)^\varepsilon)
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$$\exists x A(x) \Leftrightarrow A(\varepsilon_x A(x)) \quad \forall x A(x) \Leftrightarrow A(\varepsilon_x \neg A(x))$$

Definition

define a **mapping** ε :

$$\begin{aligned} f(t_1, \dots, t_n)^\varepsilon &= f(t_1^\varepsilon, \dots, t_n^\varepsilon) & P(t_1, \dots, t_n)^\varepsilon &= P(t_1^\varepsilon, \dots, t_n^\varepsilon) \\ x^\varepsilon &= x & (A \rightarrow B)^\varepsilon &= A^\varepsilon \rightarrow B^\varepsilon & [\varepsilon_x A(x)]^\varepsilon &= \varepsilon_x A(x)^\varepsilon \\ a^\varepsilon &= a & (A \vee B)^\varepsilon &= A^\varepsilon \vee B^\varepsilon & (\exists x A(x))^\varepsilon &= A^\varepsilon(\varepsilon_x A(x)^\varepsilon) \\ (\neg A)^\varepsilon &= \neg A^\varepsilon & (A \wedge B)^\varepsilon &= A^\varepsilon \wedge B^\varepsilon & (\forall x A(x))^\varepsilon &= A^\varepsilon(\varepsilon_x \neg A(x)^\varepsilon) \end{aligned}$$

Lemma

if π is a PC_ε -proof of A then there is an EC_ε -proof π^ε of A^ε with $\text{sz}(\pi^\varepsilon) \leq 3 \cdot \text{sz}(\pi)$ and $\text{cc}(\pi^\varepsilon) \leq \text{cc}(\pi)$

Example: Epsilon Mapping

Example

$$[\exists x(P(x) \vee \forall y Q(y))]^\varepsilon =$$

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$$\begin{aligned} [\exists x(P(x) \vee \forall y Q(y))]^\varepsilon &= \\ &= [P(x) \vee \forall y Q(y)]^\varepsilon \quad \{x \leftarrow \varepsilon_x[P(x) \vee \forall y Q(y)]^\varepsilon\} \end{aligned}$$

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 &\quad [P(x) \vee \forall y Q(y)]^\varepsilon = P(x) \vee \underbrace{Q(\varepsilon_y \neg Q(y))}_{e_1}
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 & = P(x) \vee \underbrace{Q(\varepsilon_y \neg Q(y))}_{e_1} \quad \underbrace{\{x \leftarrow \varepsilon_x[P(x) \vee \underbrace{Q(\varepsilon_y \neg Q(y))}_{e_1}]\}}_{e_2}
 \end{aligned}$$

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 &= [P(x) \vee \forall y Q(y)]^\varepsilon \quad \{x \leftarrow \varepsilon_x[P(x) \vee \forall y Q(y)]^\varepsilon\} \\
 & \quad [P(x) \vee \forall y Q(y)]^\varepsilon = P(x) \vee \underbrace{Q(\varepsilon_y \neg Q(y))}_{e_1} \\
 &= P(x) \vee \underbrace{Q(\varepsilon_y \neg Q(y))}_{e_1} \quad \{x \leftarrow \varepsilon_x[P(x) \vee \underbrace{Q(\varepsilon_y \neg Q(y))}_{e_1}]\} \\
 & \quad \underbrace{\hspace{15em}}_{e_2} \\
 &= P(\underbrace{\varepsilon_x[P(x) \vee \underbrace{Q(\varepsilon_y \neg Q(y))}_{e_1}]}_{e_2}) \vee \underbrace{Q(\varepsilon_y \neg Q(y))}_{e_1}
 \end{aligned}$$

Drinker's Paradox (Yet Again)

Example

$$\begin{array}{c}
 \frac{P(a) \Rightarrow P(a)}{} \\
 \frac{P(a) \Rightarrow P(a), \forall y P(y)}{} \\
 \Rightarrow \frac{P(a) \rightarrow \forall y P(y), P(a)}{} \\
 \Rightarrow \frac{\exists x (P(x) \rightarrow \forall y P(y)), P(a)}{} \\
 \Rightarrow \frac{\exists x (P(x) \rightarrow \forall y P(y)), \forall y P(y)}{} \\
 \frac{P(b) \Rightarrow \exists x (P(x) \rightarrow \forall y P(y)), \forall y P(y)}{} \\
 \Rightarrow \frac{\exists x (P(x) \rightarrow \forall y P(y)), P(b) \rightarrow \forall y P(y)}{} \\
 \Rightarrow \frac{\exists x (P(x) \rightarrow \forall y P(y)), \exists x (P(x) \rightarrow \forall y P(y))}{\Rightarrow \exists x (P(x) \rightarrow \forall y P(y))}
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Example

$$\begin{array}{c}
 \frac{P(a) \Rightarrow P(a)}{P(a) \Rightarrow P(a), \forall y P(y)} \\
 \Rightarrow \frac{P(a) \rightarrow \forall y P(y), P(a)}{\Rightarrow \exists x (P(x) \rightarrow \forall y P(y)), P(a)} \\
 \Rightarrow \frac{\exists x (P(x) \rightarrow \forall y P(y)), \forall y P(y)}{P(b) \Rightarrow \exists x (P(x) \rightarrow \forall y P(y)), \forall y P(y)} \\
 \Rightarrow \frac{\exists x (P(x) \rightarrow \forall y P(y)), P(b) \rightarrow \forall y P(y)}{\Rightarrow \exists x (P(x) \rightarrow \forall y P(y)), \exists x (P(x) \rightarrow \forall y P(y))} \\
 \Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y))
 \end{array}$$

where we employ

$$\begin{aligned}
 [\forall y P(y)]^\varepsilon &= P(\varepsilon_y \neg P(y)) \\
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 \frac{P(b) \Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), \forall y P(y)}{\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(b) \rightarrow \forall y P(y)} \\
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$$[P(\varepsilon_y \neg P(y)) \rightarrow P(\varepsilon_y \neg P(y))] \rightarrow [P(\underbrace{\varepsilon_x(P(x) \rightarrow P(\varepsilon_y \neg P(y)))}_{\varepsilon}) \rightarrow P(\varepsilon_y \neg P(y))]$$

Drinker's Paradox (à la Michel Parigot)

Example (cont'd)

- | | | |
|---|--|----------------|
| 1 | $P(\varepsilon_y \neg P(y)) \rightarrow P(\varepsilon_y \neg P(y))$ | TAUT |
| 2 | $(P(\varepsilon_y \neg P(y)) \rightarrow P(\varepsilon_y \neg P(y))) \rightarrow$
$\rightarrow (P(\varepsilon_x (P(x) \rightarrow P(\varepsilon_y \neg P(y)))) \rightarrow P(\varepsilon_y \neg P(y)))$ | critical axiom |
| 3 | $P(\varepsilon_x (P(x) \rightarrow P(\varepsilon_y \neg P(y)))) \rightarrow P(\varepsilon_y \neg P(y))$ | 1, 2, MP |



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Example (recall Michel's talk)

$$\frac{\Rightarrow P(a) \rightarrow P(a)}{\Rightarrow P(v) \rightarrow \forall y P(y)}$$

$$\frac{\Rightarrow P(v) \rightarrow \forall y P(y)}{\Rightarrow \exists x (P(x) \rightarrow \forall y P(y))}$$

Drinker's Paradox (à la Michel Parigot)

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Example (recall Michel's talk)

$$\frac{\Rightarrow P(\varepsilon_y \neg P(y)) \rightarrow P(\varepsilon_y \neg P(y))}{\Rightarrow P(\varepsilon_y \neg P(y)) \rightarrow \forall y P(y)}$$

$$\frac{\quad}{\Rightarrow \exists x (P(x) \rightarrow \forall y P(y))}$$

Proof of the Lemma

Proof

- we show \forall proofs $\pi: A_1, \dots, A_n$
 \exists proof π^ε containing $A_1^\varepsilon, \dots, A_n^\varepsilon$ (+ extra formulas)
- we use by induction on n

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- base case is trivial and if $A_n =: A$ is a propositional tautology, A^ε is also a tautology
- **Case** A an instance of a quantifier axiom; suppose
 $A = A(t) \rightarrow \exists x A(x)$; hence

$$[A(t) \rightarrow \exists x A(x)]^\varepsilon = A^\varepsilon(t^\varepsilon) \rightarrow A^\varepsilon(\varepsilon_x A(x)^\varepsilon)$$

the latter is a critical axiom

Proof of the Lemma

Proof

- we show \forall proofs $\pi: A_1, \dots, A_n$
 \exists proof π^ε containing $A_1^\varepsilon, \dots, A_n^\varepsilon$ (+ extra formulas)
- we use by induction on n
- base case is trivial and if $A_n =: A$ is a propositional tautology, A^ε is also a tautology
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- **Case** A follows by modus ponens from A_i and $A_j \equiv A_i \rightarrow A$
 applying IH there exists a proof π^* containing A_i^ε and $A_i^\varepsilon \rightarrow A_j^\varepsilon$; we
 add A^ε to π^*

Proof (cont'd).

- **Case A** follows by generalisation; i.e. $A = B \rightarrow \forall x C(x)$ and there exists $A_i = B \rightarrow C(a)$; a eigenvariable
by IH there exists a proof π^* containing $A_i^\varepsilon \equiv B^\varepsilon \rightarrow C(a)^\varepsilon$; replacing the eigenvariable a by $\varepsilon_x \neg A^\varepsilon(x)$ results in a proof containing

$$B^\varepsilon \rightarrow A^\varepsilon(\varepsilon_x \neg A^\varepsilon(x)) = [B \rightarrow \forall x C(x)]^\varepsilon$$

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Lemma (Embedding Lemma)

if π is a PC_ε -proof of A then there is an EC_ε -proof π^ε of A^ε with $\text{sz}(\pi^\varepsilon) \leq 3 \cdot \text{sz}(\pi)$ and $\text{cc}(\pi^\varepsilon) \leq \text{cc}(\pi)$

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Quiz

Question

the proof of the embedding lemma is wrong; can you spot the mistake?



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the proof of the embedding lemma is wrong; can you spot the mistake?

Answer

the application of IH in the generalisation case requires more work^a

^apaper by M., Zach contains the presented proof; bug was spotted by Michel Parigot, thank!

The First Epsilon Theorem

Theorem

suppose $E(e_1, \dots, e_m)$ is a quantifier-free formula containing only the ε -terms s_1, \dots, s_m , and

$$EC_\varepsilon \vdash_\pi E(s_1, \dots, s_m),$$

then there are ε -free terms t_j^i such that

$$EC \vdash \bigvee_{i=1}^n E(t_1^i, \dots, t_m^i)$$

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number of instances independent of # of propositional inferences

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Observations

- the upper bound on the length of the Herbrand disjunction depends only on the **critical count** of the initial proof
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Question

what about lower-bounds of the ε -elimination procedure

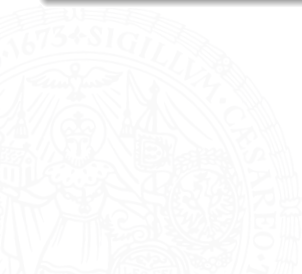
Lower Bounds

Definition

- an **V-expansion** (of $E \equiv E(s_1, \dots, s_m)$) is a finite disjunction

$$E' \equiv E_1 \vee \dots \vee E_l$$

$$E_i \equiv E(s_1^i, \dots, s_m^i) \text{ for terms } s_j^i$$



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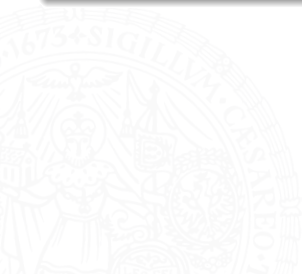
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Theorem

there is a sequence of formulas E_k so that

- for each k , \exists PC_ε -proof π_k of E_k with $\text{cc}(\pi_k) \leq c \cdot k$, but
- $\text{HC}(E_k) \geq 2_k^1$.

Proof Sketch of Part 1

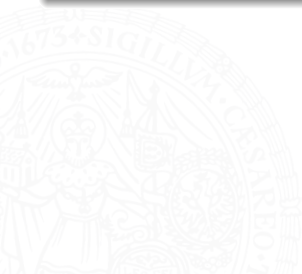
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$$\text{Hyp}_1 := \forall x R(x, 0, S(x))$$

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$$C_k := \exists z_k \dots \exists z_0 (R(0, 0, z_k) \wedge R(0, z_k, z_{k-1}) \wedge \dots \wedge R(0, z_1, z_0))$$

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for every k , $\text{PC}_\varepsilon \vdash_{\pi_k} E_k$, where $\text{cc}(\pi_k) = c \cdot k$

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this establishes part one of the theorem

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consider Herbrand **sequents** of the sequent $\text{Hyp}_1, \text{Hyp}_2 \Rightarrow C_k$

- 1 each of these sequents has the form $\Gamma_1, \Gamma_2 \Rightarrow \Delta$



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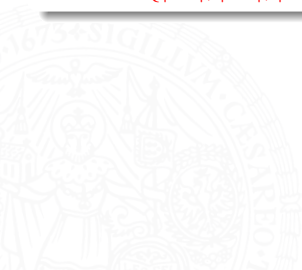


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Theorem

If A is a formula of $L(PC)$ and $PC_\varepsilon \vdash A$, then $PC \vdash A$.



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- 1 we only treated the case **without** equality
- 2 ε -theorems and Herbrand's theorem: proof theory without sequents
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Future Work

finally sort out the case **with** equality:

- 1 equality is represented by (ε -)identity schema
- 2 known method for ε -elimination approximates possible size of atom formulas
- 3 destroys exclusive dependency of length of Herbrand disjunction on critical count

A Big Thank You to **Alessio**, **Anupam**,
Lutz, **Paola**, and **Willem** for this Exciting
Workshop!

... and Thanks All of You for Your
Attention!

