KBOs, Ordinals, Subrecursive Hierarchies and All That*

Georg Moser
Institute of Computer Science
University of Innsbruck, Austria
georg.moser@uibk.ac.at

June 23, 2014

We study the complexity of term rewrite systems compatible with the Knuth-Bendix order, if the signature of the rewrite system is potentially infinite. We show that the known bounds on the derivation height are essentially preserved, if the rewrite system fulfils some mild conditions. This allows us to obtain bounds on the derivational height of non simply terminating rewrite systems. As a corollary, we re-establish an essentially optimal 2-recursive upper bound on the derivational complexity of finite rewrite systems compatible with a Knuth-Bendix order. Furthermore we link our main result to results on generalised Knuth-Bendix orders and to recent results on transfinite Knuth-Bendix orders.

to Alex on this momentous occasion

1 Introduction

Studies into the *complexity* of term rewrite systems (TRSs for short) are concerned with the conception of TRSs as computation model and aim to measure the provided programs with respect to their efficiency. Conceptually one typically limits to terminating TRSs for such a quest, although almost all known techniques to assess the complexity of a given TRS entail termination of the input TRS.

Several notions to assess the complexity of a terminating TRS have been proposed in the literature, compare [7, 16, 9, 14]. In this paper we are mainly concerned with the *derivational complexity* and the *runtime complexity* of terminating TRS. Here the *derivational complexity function* (denoted as $\mathsf{dc}_{\mathcal{R}}$) with respect to a terminating TRS

* This work is partly supported by FWF (Austrian Science Fund) project I-603-N18.

$\mathcal{R}$ relates the maximal derivation height to the size of the initial term [16]. On the other hand the *runtime complexity function* (denoted as $\mathsf{rc}_{\mathcal{R}}$) with respect to $\mathcal{R}$ relates the length of the longest derivation sequence to the size of the initial term, where the arguments are supposed to be in normal form [14]. The motivation for the second (more restricted) notion is that often one is not interested into the derivation height of a general term, but only of a function.

The study of the complexity of TRS is intrinsically related to the study of the *termination* of a TRS and often it has been possible to yields an upper bound on the complexity of a rewrite system $\mathcal{R}$ from an analysis of the termination proof of $\mathcal{R}$. We mention two recent results that show different aspects of interest (see [29] for further references). For one we mention that the *dependency pair framework* [12] in conjunction with the *subterm criterion* [13] characterises the multiple-recursive functions, cf. [30, 34]. Here (derivational) complexity analysis yields qualitative information on termination analysis: rewrite systems that admit derivational complexities beyond multiple-recursive cannot be handled by the mentioned termination criteria. One the other hand, variants of interpretations into (vectors) of natural numbers, originally studied for termination, sometimes yield a derivational and runtime complexity analysis that is precise upto the degree of the polynomial that bounds the complexity, see for example [27]. Hence, termination techniques can be refined to yield precise bounds on the complexity of rewrite systems.

In this paper we are concerned with the complexity of TRSs whose termination can be shown with (variants) of *Knuth-Bendix orders* (*KBOs* for short). Our main result is concerned with the analysis of *standard* KBOs, where we show that earlier results by Lepper [21] yielding an optimal 2-recursive bound on the derivational (and runtime) complexity for TRS over finite signatures can be generalised to provide bounds on the *derivation height* for TRS over infinite signatures (Theorem 7.1). In this context we also remark on the difficulty to obtain bounds on the *derivational* (or *runtime*) *complexity functions* for infinite signatures.

Then we apply this result to *general* KBOs (*GKBOs* for short) [25] to obtain a 2-recursive upper bound on the *derivation height* for TRSs over infinite signatures that are compatible with GKBOs (Theorem 9.1). GKBOs generalise KBOs in that the weight function employed in KBOs is replaced by a weakly-monotone simple algebra $(\mathcal{A}, \sqsupset)$. As the complexity of any TRS $\mathcal{R}$ compatible with a GKBO $\succ_{\mathsf{gkbo}}$ is dependent on the complexity induced by $\sqsupset$, this result becomes only directly applicable if further restrictions are made. In particular we can restrict to finite signatures to establish that compatibility of a TRS $\mathcal{R}$ with $\succ_{\mathsf{gkbo}}$ induces an incredible derivational complexity: $\mathsf{dc}_{\mathcal{R}}$ is dominated by $\mathsf{H}_{\vartheta\Omega^\omega}$, where $(\mathsf{H}_\alpha)_{\alpha\in\mathsf{On}}$ denotes the *Hardy hierarchy* and $\vartheta\Omega^\omega$ the *small Veblen number* (Theorem 9.2). Surprisingly this enormous upper bound is optimal. At the same time we prove that the order type of any $\succ_{\mathsf{gkbo}}$ (over a finite signature) is very modest: given that the auxiliary order $\sqsupset$ is finitely branching, the order type of $\succ_{\mathsf{gkbo}}$ is ω (Lemma 9.2).

Finally, we study *transfinite* KBOs (*TKBOs* for short) [23], where transfinite ordinal weights may be used as weights. We re-prove a result by Winkler et al. [40] that any *finite* TRS compatible with a TKBO is compatible with a KBO. For this we employ the *slow-growing hierarchy* $(\mathsf{G}_\alpha)_{\alpha\in\mathsf{On}}$. Based on this result we obtain a derivational (and runtime) complexity analysis of finite TRSs compatible with TKBOs (Theorem 10.1).

The above results implicitly point to a intricate connection between the *order type* of a reduction order $\succ$ employed to show termination of $\mathcal{R}$ and the *complexity* of $\mathcal{R}$. Cichon [8] conjectured that the slow-growing hierarchy should provide such a link: if α is the order type of $\succ$, G_α should eventually dominate $\mathsf{dc}_\mathcal{R}$. Touzet invalidated this conjecture by encoding suitable large parts of the Hydra battle [18] as TRSs. A compact counter-argument was given by Lepper: the order type of any KBO $\succ_{\mathsf{kbo}}$ is ω^ω, but $\mathsf{dc}_\mathcal{R}$ belongs to $\mathsf{Ack}(O(n), 0)$, cf. [21] and Corollary 7.3. This result may suggest that a fast-growing hierarchy provides the link between order type of a reduction order and the complexity of a rewrite system. In particular Touzet conjectured: "the Hardy hierarchy is the right tool for connecting derivation length and order type" [38]. This conjecture is partly emphasised by the observation that the below presented proof of Lepper's result exploits the order type of $\succ_{\mathsf{kbo}}$ *and* the connection of the Ackermann function to the Hardy hierarchy. Despite of this, the conjecture does not hold in general. This time *generalised* KBO provides a counter-example. As mentioned above the order type of a GKBO $\succ_{\mathsf{gkbo}}$ may be ω, while this GKBO by induce a derivation complexity that majorises any H_α for $\alpha < \vartheta\Omega^\omega$.

The remainder of this paper is organised as follows. In the next two section we recall basic notions and starting points of this paper. In Sections 4–7 we develop our first main result on the complexity of KBO. Section 8 yields an application of this result to the complexity analysis of TRSs whose termination has been shown by semantic labelling. These results have already been presented in [28] and appear here in polished form. Section 9 establish the results on GKBOs and Section 10 is concerned with TKBOs.

2 Preliminaries

We denote by $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, \dots\}$. Let R be a binary relation. The transitive closure of R is denoted by R^+ and its transitive and reflexive closure by R^*. For a binary relation R, we frequently write $a\ R\ b$ instead of $(a, b) \in R$. Composition of binary relations R and S is denoted by $R \cdot S$, and defined in the usual way. For $n \in \mathbb{N}$ we denote by R^n the *n-fold composition of* R. Similarly, we write $f^n(\cdot)$ for the n-fold iteration of function f.

A binary relation R is *well-founded* if there exists no infinite chain $a_0, a_1, \dots$ with $a_i\ R\ a_{i+1}$ for all $i \in \mathbb{N}$. Moreover, we say that R is well-founded on a set A if there exists no such infinite chain with $a_0 \in A$. The relation R is *finitely branching* if for all elements a, the set $\{b \mid a\ R\ b\}$ is finite. A *proper order* is an irreflexive and transitive binary relation. A *preorder* is a reflexive and transitive binary relation. An *equivalence relation* is reflexive, symmetric and transitive. Every preorder $\succcurlyeq$ induces a proper order $\succ$, namely $a \succ b$ if and only if $a \succcurlyeq b$ and not $b \succcurlyeq a$, and an equivalence relation $\sim$: $a \sim b$ iff $a \succcurlyeq b$ and $b \succcurlyeq a$. A proper order is called *linear* (or *total*) on A if for all $a, b \in A$, a different from b, a and b are comparable by $\succ$. A proper order $\succ$ on a set A is *well-founded* if there exists no infinite descending sequence $a_1 \succ a_2 \succ \cdots$ of elements of A and $\succ$ is a *well partial order* if every order that extends $\succ$ is well-founded. The *lexicographic product* of n proper orders $\succ_i$ on A_i is defined as the proper order $\succ_{\mathrm{lex}}^n$ where $(a_1, \ldots, a_n) \succ_{\mathrm{lex}}^n (b_1, \ldots, b_n)$ holds if there exists $i \in \{1, \ldots, n\}$ such that for all $j \in \{0, \ldots, i-1\}$ $a_j = b_j$ and $a_i \succ_i b_i$. The *lexicographic order* on A^* based on a proper order $\succ$ on A is defined as follows: $a \succ_{\mathrm{lex}}^* b$ holds if either (i) $|a| > |b|$, or (ii) $|a| = |b| =: n$ and $a \succ_{\mathrm{lex}}^n b$. (Note that $\succ_{\mathrm{lex}}^n$ denotes the n^{th} lexicographic product of $\succ$.)

2.1 Term Rewriting

We assume familiarity with term rewriting [5, 37] but review basic concepts and notations from rewriting.

Let $\mathcal{V}$ denote a countably infinite set of variables and $\mathcal{F}$ a signature. We assume that $\mathcal{F}$ contains at least one constant. The set of terms over $\mathcal{F}$ and $\mathcal{V}$ is denoted as $\mathcal{T}(\mathcal{F}, \mathcal{V})$. The set of ground terms is written as $\mathcal{T}(\mathcal{F})$. The top symbol of a term t is called its *root symbol* and denoted as $\mathsf{rt}(t)$. The (proper) subterm relation is denoted as $\trianglelefteq$ ($\triangleleft$). $\mathcal{V}\mathsf{ar}(t)$ denotes the set of variables occurring in a term t. The *size* $\mathsf{sz}(t)$ of a term is defined as the number of symbols in t:

$$\mathsf{sz}(t) := \begin{cases} 1 & \text{if } t \text{ is a variable ,} \\ 1 + \sum_{1 \leqslant i \leqslant n} \mathsf{sz}(t_i) & \text{if } t = f(t_1, \ldots, t_n) \text{ .} \end{cases}$$

The *depth* of a term t is denoted as $\mathsf{dp}(t)$. The number of occurrences of a symbol $a \in \mathcal{F} \cup \mathcal{V}$ in t is denoted as $|t|_a$.

Let $\succ$ denote a proper order on the signature $\mathcal{F}$, called *precedence*. The *rank* of a function symbol is inductively defined by $\mathsf{rk}(f) = \max\{0\} \cup \{1 + \mathsf{rk}(g) \mid f \succ g\}$. A *rewrite rule* is a pair (l, r) of terms, in notation $l \to r$, such that l is not a variable and all variables in r occur also in l. Here l is called the *left-hand*, and r the *right-hand side* of $l \to r$. A *term rewrite system* (*TRS* for short) $\mathcal{R}$ over $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is a set of *rewrite rules*. Let $\mathcal{R}$ be a TRS and $\mathcal{F}$ a signature. A function symbol $f \in \mathcal{F}$ that occurs as root symbol of a left-hand side of $\mathcal{R}$ is called *defined*; otherwise f is called a *constructor* symbol. We decompose $\mathcal{F}$ into two sets $\mathcal{D}$ and $\mathcal{C}$ such that the defined symbols are collected in $\mathcal{D}$, while the constructor symbols are collected in $\mathcal{C}$; $\mathcal{F} = \mathcal{D} \cup \mathcal{C}$. A relation on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is a *rewrite relation* if it is closed under contexts and closed under substitutions. The smallest rewrite relation that contains $\mathcal{R}$ is denoted by $\to_{\mathcal{R}}$ (or $\to$ if the TRS $\mathcal{R}$ is clear from context). A rewrite relation that is also a proper order is called *rewrite order*. A well-founded rewrite order is called *reduction order*.

A term $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is called a *normal form* if there is no $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ such that $s \to t$. Whenever t is a normal form of $\mathcal{R}$ we write $s \xrightarrow{!}_{\mathcal{R}} t$ for $s \xrightarrow{*}_{\mathcal{R}} t$. The TRS $\mathcal{R}$ is *terminating* if no infinite rewrite sequence exists. A TRS $\mathcal{R}$ and a proper order $\succ$ are *compatible* if $\mathcal{R}$ is contained in $\succ$, denoted as $\mathcal{R} \subseteq \succ$. We also say that $\mathcal{R}$ is compatible with $\succ$ or vice versa. A TRS $\mathcal{R}$ is terminating iff it is compatible with a reduction order $\succ$. A TRS $\mathcal{R}$ is called *confluent* if for all $s, t_1, t_2 \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ with $s \xrightarrow{*} t_1$ and $s \xrightarrow{*} t_2$ there exists a term u such that $t_1 \xrightarrow{*} u$ and $t_2 \xrightarrow{*} u$.

2.2 Termination

We assume at least nodding acquaintance with the basics of termination analysis in rewriting, cf. [37, Chapter 6] or [5]. We briefly review the use of the *semantic labelling* technique for showing termination.

Let $(\mathcal{A}, >)$ denote a well-founded weakly monotone $\mathcal{F}$-algebra; $(\mathcal{A}, >)$ consists of a carrier A, interpretations $f_\mathcal{A}$ for each function symbol in $\mathcal{F}$, and a well-founded proper order $>$ on A such that every $f_\mathcal{A}$ is weakly monotone in all arguments. We define a quasi-order $\geqslant_\mathcal{A}$: $s \geqslant_\mathcal{A} t$ if for all assignments $\alpha\colon \mathcal{V} \to A$: $[\alpha]_\mathcal{A}(s) \geqslant [\alpha]_\mathcal{A}(t)$. Here $\geqslant$ denotes the reflexive closure of $>$ and $[\alpha]_\mathcal{A}(\cdot)$ denotes the evaluation function with respect to $\mathcal{A}$. The algebra $(\mathcal{A}, >)$ is a *quasi-model* of a TRS $\mathcal{R}$, if $\mathcal{R} \subseteq \geqslant_\mathcal{A}$.

A *labelling* ℓ for $\mathcal{A}$ consists of a set of labels L_f together with mappings $\ell_f\colon A^n \to L_f$ for every $f \in \mathcal{F}$, f n-ary. A labelling is called *weakly monotone* if all labelling functions ℓ_f are weakly monotone in all arguments. The labelled signature $\mathcal{F}_{\text{lab}}$ consists of n-ary functions symbols f_a for every $f \in \mathcal{F}$, $a \in L_f$, together with all $f \in \mathcal{F}$, such that $L_f = \varnothing$. The TRS $\mathcal{D}ec$ consists of all rules $f_{a+1}(x_1, \ldots, x_n) \to f_a(x_1, \ldots, x_n)$ for all $f \in \mathcal{F}$. The x_i denote pairwise different variables. For every assignment α, we inductively define a mapping $\mathsf{lab}_\alpha\colon\ \mathcal{T}(\mathcal{F}, \mathcal{V}) \to \mathcal{T}(\mathcal{F}_{\text{lab}}, \mathcal{V})$:

$$\mathsf{lab}_\alpha(t) := \begin{cases} t & \text{if } t \in \mathcal{V}\ , \\ f(\mathsf{lab}_\alpha(t_1), \ldots, \mathsf{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f = \varnothing\ , \\ f_a(\mathsf{lab}_\alpha(t_1), \ldots, \mathsf{lab}_\alpha(t_n)) & \text{otherwise}\ . \end{cases}$$

The label a in the last case is defined as $l_f([\alpha]_\mathcal{A}(t_1), \ldots, [\alpha]_\mathcal{A}(t_n))$. The *labelled* TRS $\mathcal{R}_{\text{lab}}$ over $\mathcal{F}_{\text{lab}}$ is defined as $\mathcal{R}_{\text{lab}} := \{\mathsf{lab}_\alpha(l) \to \mathsf{lab}_\alpha(r) \mid l \to r \in \mathcal{R} \text{ and } \alpha \text{ an assignment}\}$.

Proposition 2.1 ([42])**.** *Let $\mathcal{R}$ be a TRS, $(\mathcal{A}, >)$ a well-founded quasi-model for $\mathcal{R}$, and ℓ a weakly monotone labelling for $(\mathcal{A}, >)$. Then $\mathcal{R}$ is terminating iff $\mathcal{R}_{\text{lab}} \cup \mathcal{D}ec$ is terminating.*

The proof of the proposition uses the following lemma.

Lemma 2.1. *Let $\mathcal{R}$ be a TRS, $(\mathcal{A}, >)$ a quasi-model of $\mathcal{R}$, and ℓ a weakly monotone labelling for $(\mathcal{A}, >)$. If $s \to_\mathcal{R} t$, then $\mathsf{lab}_\alpha(s) \xrightarrow{*}_{\mathcal{D}ec} \cdot \to_{\mathcal{R}_{\text{lab}}} \mathsf{lab}_\alpha(t)$ for all assignments α.*

The *derivation height* of a term s with respect to a well-founded, finitely branching relation $\to$ is defined as follows:

$$\mathsf{dheight}(s, \to) := \max\{n \mid \exists t\ s \to^n t\}\ .$$

From Lemma 2.1, we obtain the next corollary.

Corollary 2.1. *Let $\mathcal{R}$ be a TRS, $(\mathcal{A}, >)$ a well-founded quasi-model for $\mathcal{R}$, and ℓ a weakly monotone labelling for $(\mathcal{A}, >)$. Then for all terms t*

$$\mathsf{dheight}(t, \to_\mathcal{R}) \leqslant \mathsf{dheight}(t, \to_{\mathcal{R}_{\text{lab}} \cup \mathcal{D}ec})\ .$$

2.3 Complexity of Rewriting

Let $\mathcal{R}$ be a TRS and T be a set of terms. The *complexity function with respect to a relation $\to$ on T* is defined as follows:

$$\mathsf{comp}(n, T, \to) = \max\{\mathsf{dheight}(t, \to) \mid t \in T \text{ and } \mathsf{sz}(t) \leqslant n\} \ .$$

We call a term $t = f(t_1, \ldots, t_n)$ *basic* or *constructor based* if $f \in \mathcal{D}$ and $t_i \in \mathcal{T}(\mathcal{C}, \mathcal{V})$ for all $1 \leqslant i \leqslant n$. Let $\mathcal{T}_{\mathsf{b}}$ denote the set of *basic* terms.

Definition 2.1. Let $\mathcal{R}$ be a TRS. We define the *derivational complexity function* $\mathsf{dc}_{\mathcal{R}}(n)$ and the *runtime complexity function* $\mathsf{rc}_{\mathcal{R}}(n)$ as follows:

$$\begin{aligned} \mathsf{dc}_{\mathcal{R}}(n) &:= \mathsf{comp}(n, \mathcal{T}(\mathcal{F}, \mathcal{V}), \to_{\mathcal{R}}) \\ \mathsf{rc}_{\mathcal{R}}(n) &:= \mathsf{comp}(n, \mathcal{T}_{\mathsf{b}}, \to_{\mathcal{R}}) \ . \end{aligned}$$

Note that the above complexity functions need not be defined. In particular the rewrite relation $\to_{\mathcal{R}}$ need not be well-founded *and* finitely branching. The next example illustrates a difference between derivational complexity and runtime complexity.

Example 2.1. Consider the following TRS $\mathcal{R}_1$[1]

$$\begin{aligned} &1\colon & x - \mathsf{0} &\to x & \quad &3\colon & \mathsf{0} \div \mathsf{s}(y) &\to \mathsf{0} \\ &2\colon & \mathsf{s}(x) - \mathsf{s}(y) &\to x - y & \quad &4\colon & \mathsf{s}(x) \div \mathsf{s}(y) &\to \mathsf{s}((x - y) \div \mathsf{s}(y)) \ . \end{aligned}$$

Although the functions *computed* by $\mathcal{R}_1$ are obviously feasible this is not reflected in the derivational complexity of $\mathcal{R}_1$. Consider rule 4, which we abbreviate as $C[y] \to D[y, y]$. Since the maximal derivation height starting with $C^n[y]$ equals 2^{n-1} for all $n > 0$, $\mathcal{R}_1$ admits (at least) exponential derivational complexity. In general any duplicating TRS admits (at least) exponential derivational complexity.

For completeness of the exposition we clarify what it means that a TRS *computes* a function. One subtlety here is that TRS a typically not confluent, hence the resulting computation is nondeterministic. However, for our context it suffices to restrict to *confluent* TRS; see [2] for the general definition.

Definition 2.2. Let $\mathcal{R}$ be a confluent TRS; for each n-ary defined function symbol $f \in \mathcal{F}$ we define the function $[\![f]\!]\colon \mathcal{T}(\mathcal{C}, \mathcal{V})^n \to \mathcal{T}(\mathcal{C}, \mathcal{V})$ as follows:

$$[\![f]\!](v_1, \ldots, v_n) = w \quad \text{iff} \quad f(v_1, \ldots, v_n) \xrightarrow{!}_{\mathcal{R}} w \ .$$

We call $[\![f]\!]$ the *function defined by f in $\mathcal{R}$* and say that $\mathcal{R}$ *computes* the function associated with $[\![f]\!]$.

We remark that the runtime complexity of a TRS forms an *invariant cost model*, that is, the runtime complexity with respect to a rewrite system $\mathcal{R}$, is polynomial in the complexity of the function computed by $\mathcal{R}$, cf. [2, 4].

[1] This is Example 3.1 in Arts and Giesl's collection of TRSs [1].

2.4 Set-Theory

We briefly review a few basic concepts from set-theory in particular ordinals, see [17]. We write $>$ to denote the well-ordering of ordinals. Any ordinal $\alpha \neq 0$, smaller than ϵ_0, can uniquely be represented by its *Cantor Normal Form* (*CNF* for short):

$$\omega^{\alpha_1} n_1 + \cdots + \omega^{\alpha_k} n_k \qquad \text{with } \alpha_1 > \cdots > \alpha_k \ .$$

To each well-founded proper order $\succ$ on a set A we can associate a (set-theoretic) ordinal, its *order type*. First we associate an ordinal to each element a of A by setting $\mathsf{otype}_{\succ}(a) := \sup\{\mathsf{otype}_{\succ}(b) + 1 \colon b \in A \text{ and } b \succ a\}$. The *order type* of $\succ$, denoted by $\mathsf{otype}(\succ)$, is the supremum of $\mathsf{otype}_{\succ}(a) + 1$ with $a \in A$. For two proper orders $\succ$ and $\succ'$ on A and A', respectively, a mapping $\mathsf{o} \colon A \to A'$ *embeds* $\succ$ into $\succ'$ if for all $p, q \in A$, $p \succ q$ implies $\mathsf{o}(p) \succ' \mathsf{o}(q)$. Such a mapping is an *order-isomorphism* if it is bijective and the proper orders $\succ$ and $\succ'$ are linear .

We recall that the order type of the lexicographic product of well-founded proper orders $\succ_i$ and A_i is the reverse product of the order types of the base orders:

$$\mathsf{otype}(\succ_{\mathrm{lex}}^{n}) = \mathsf{otype}(\succ_n) \cdots \mathsf{otype}(\succ_1) \ ,$$

where we employ standard multiplication of ordinals [17].

Lemma 2.2. *Let $\succ$ be well-founded and let $\mathsf{otype}(\succ) \geqslant \omega$. Then $\mathsf{otype}(\succ_{\mathrm{lex}}^{*}) = \mathsf{otype}(\succ)^{\omega}$.*

Proof. We put $\alpha = \mathsf{otype}(\succ) \geqslant \omega$ and consider the n^{th} lexicographic product of $\succ$. By the above observation we obtain that $\mathsf{otype}(\succ_{\mathrm{lex}}^{n}) = \alpha^n$. Furthermore we have for all n:

$$\alpha^n + \alpha^{n+1} = \alpha^n \cdot (1 + \alpha) = \alpha^n \cdot \alpha = \alpha^{n+1} \ .$$

As $\succ_{\mathrm{lex}}^{*}$ amounts to an infinite concatenation of $\succ_{\mathrm{lex}}^{n}$ we have:

$$\begin{aligned}
\mathsf{otype}(\succ_{\mathrm{lex}}^{*}) &= \sup_{n<\omega}(\mathsf{otype}(\succ_{\mathrm{lex}}^{1}) + \cdots + \mathsf{otype}(\succ_{\mathrm{lex}}^{n})) \\
&= \sup_{n<\omega}(\alpha^1 + \cdots + \alpha^n) = \sup_{n<\omega} \alpha^n = \alpha^{\omega} \ ,
\end{aligned}$$

where we have employed that the order type of the concatenation of proper orders is the sum of the order types of the base orders and in the very last step the definition of ordinal exponentiation. □

3 Knuth-Bendix Orders

A *weight function* for $\mathcal{F}$ is a pair (w, w_0) consisting of a function $\mathsf{w} \colon \mathcal{F} \to \mathbb{N}$ and a minimal weight $w_0 \in \mathbb{N}$, $w_0 > 0$ such that $\mathsf{w}(c) \geqslant w_0$ if c is a constant. A weight function (w, w_0) is called *admissible* for a precedence $\succ$ if $f \succ g$ for all $g \in \mathcal{F}$ different from f, when f is unary with $\mathsf{w}(f) = 0$. The function symbol f (if present) is called *special* and denoted by i. The *weight* of a term t, denoted as $\mathsf{w}(t)$ is defined as follows:

$$\mathsf{w}(t) := \begin{cases} w_0 & t \text{ is a variable} \\ \mathsf{w}(f) + \mathsf{w}(t_1) + \cdots + \mathsf{w}(t_n) & \text{if } t = f(t_1, \ldots, t_n) \ . \end{cases}$$

We recall the standard definition of the *Knuth-Bendix Orders* (*KBOs* for short). Let (w, w_0) denote an admissible weight function for $\mathcal{F}$ and let $\succ$ denote a precedence on $\mathcal{F}$. The *Knuth-Bendix order* $\succ_{\mathsf{kbo}}$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is inductively defined as follows: $s \succ_{\mathsf{kbo}} t$ if $\forall$ x: $|s|_x \geqslant |t|_x$ and

1. $\mathsf{w}(s) > \mathsf{w}(t)$, or
2. $\mathsf{w}(s) = \mathsf{w}(t)$, and one of the following alternatives holds:
   a) t is a variable, $s = f^k(t)$, $k > 0$,
   b) $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$, and $f \succ g$.,
   c) $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, and there exists $i \in \{1, \ldots, n\}$ such that $s_1 = t_1, \ldots, s_{i-1} = t_{i-1}$ and $s_i \succ_{\mathsf{kbo}} t_i$.

The above definition is a slight (but equivalent) re-formulation of the original definition by Knuth and Bendix. However, we do not presuppose *finiteness* of the signature as is typically the case, cf. [5], see [35, 41] for the exception.

It is well-known that KBO (over finite signatures) is a reduction order as KBO fulfils the subterm property and thus (the finitary version of) Kruskal's Theorem is applicable. In particular KBO is a *simplification order*, that is, $\succ_{\mathsf{kbo}}$ extends the subterm relation.

For infinite signatures the situation is less clear, as then the subterm property is no longer sufficient to yield that KBO is well-founded. Instead one has to verify that KBO extends the *homeomorphic embedding relation*. Let $\succ$ be a well partial order. The TRS $\mathcal{E}\mathsf{mb}(\mathcal{F}, \succ)$ consists of the following rewrite rules for all $f, g \in \mathcal{F}$: $f(x_1, \ldots, x_n) \to x_i$ and $f(x_1, \ldots, x_n) \to g(x_{i_1}, \ldots, x_{i_n})$, where $1 \leqslant i_1 < \cdots i_m \leqslant n$ and $f \succ g$ in the second rule. We write $s \succ_{\mathsf{emb}} t$ if $s \xrightarrow{*}_{\mathcal{E}\mathsf{mb}(\mathcal{F},\succ]} t$ holds. One can show that for any KBO $\succ_{\mathsf{kbo}}$ induced by a well-founded precedence $\succ$, $\succ_{\mathsf{kbo}}$ is a reduction order. For this observe that $\succ$ can be extended to a total order $\succ'$, which is a well partial order by definition. Furthermore from Kruskal's Theorem (in its general form) it follows that $\succ'_{\mathsf{kbo}}$ is a simplification order (for arbitrary signatures) as it extends $\succ'_{\mathsf{emb}}$. Finally $\succ_{\mathsf{kbo}} \subseteq \succ'_{\mathsf{kbo}}$ follows from the incrementality of KBO.

Proper extensions of the above definition have been studied that use a quasi-order $\succcurlyeq$ instead of the proper order $\succ$ [35, 41]. The order $\succcurlyeq$ is usually called *quasi-precedence*. We remark that Kruskal's theorem is no longer applicable to formulations of KBO over infinite signatures *and* quasi-precedences. However it is not difficult to establish well-foundedness of such a variant [36]. It is an open problem whether our results can be generalised to KBOs over quasi-precedences. In the literature *real-valued* KBOs are studied as well, cf. [24, 10]. However, as established in [19] (see also [41]) any TRS compatible to a real-valued KBO is also compatible to a KBO in the above sense. We remark that this holds despite the fact that real-valued and number-valued KBOs are clearly not equivalent cf. [21].

In the following, we will not use the above definition of KBO, but rather consider a variant tailored to our later purposes. The variant is taken from [21]. We write $s = \mathsf{i}^a s'$ if $s = \mathsf{i}^a(s')$ and the root symbol of s' is distinct from the special symbol i.

Definition 3.1. Let (w, w_0) denote an admissible weight function for $\mathcal{F}$ and let $\succ$ denote a precedence on $\mathcal{F}$. Then $s \succ_{\mathsf{kbo2}} t$ holds if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and

1. $\mathsf{w}(s) > \mathsf{w}(t)$, or
2. $\mathsf{w}(s) = \mathsf{w}(t)$, $s = \mathsf{i}^a s'$, $t = \mathsf{i}^b t'$, and one of the following cases holds.
   a) $a > b$, or
   b) $a = b$, $s' = g(s_1, \ldots, s_n)$, $t' = h(t_1, \ldots, t_m)$, and $g \succ h$, or
   c) $a = b$, $t' = f(t_1, \ldots, t_n)$ and there exists $i \in \{1, \ldots, n\}$ such that $s_1 = t_1, \ldots, s_{i-1} = t_{i-1}$ and $s_i \succ_{\mathsf{kbo2}} t_i$.

The next lemma is not difficult to see; for a formal proof see [21].

Lemma 3.1. *The order $\succ_{\mathsf{kbo}}$ and $\succ_{\mathsf{kbo2}}$ coincide on $\mathcal{T}(\mathcal{F}, \mathcal{V})$, in particular $\succ_{\mathsf{kbo2}}$ is a reduction order.*

To simplify notation in the following we write $\succ_{\mathsf{kbo}}$ interchangingly for $\succ_{\mathsf{kbo}}$ and $\succ_{\mathsf{kbo2}}$, respectively. We conclude this section by exemplifying the definition of KBO for infinitary signatures.

Example 3.1. Consider the TRS $\mathcal{R}_2$ consisting of the following (schematic) rewrite rules:

$$\begin{array}{llll} 1.n\colon & \mathsf{f}_{n+1}(\mathsf{h}(x)) \to \mathsf{f}_n(\mathsf{i}(x)) & 2.n\colon & \mathsf{f}_{n+1}(x) \to \mathsf{f}_n(x) \\ 3\colon & \mathsf{h}(\mathsf{a}) \to \mathsf{b} & 4\colon & \mathsf{g}(\mathsf{j}(x)) \to \mathsf{g}(\mathsf{h}(x)) \\ 5\colon & \mathsf{j}(\mathsf{a}) \to \mathsf{b}\ . & & \end{array}$$

Furthermore consider the KBO $\succ_{\mathsf{kbo}}$ induced by the (admissible) weight function that sets $\mathsf{w}(f) := 1$ for all function symbols f and the following precedence $\succ$:

$$\mathsf{f}_{n+1} \succ \mathsf{f}_n \succ \ldots \succ \mathsf{f}_0 \succ \mathsf{j} \succ \mathsf{h} \succ \mathsf{g} \succ \mathsf{a} \succ \mathsf{b}\ .$$

It is easy to see that $\mathcal{R}_2 \subseteq \succ_{\mathsf{kbo}}$. Thus termination of $\mathcal{R}_2$ is guaranteed.

4 Exploiting the Order-Type of KBOs

We write $\mathbb{N}^*$ to denote the set of finite sequences of natural numbers. Let $p \in \mathbb{N}^*$, we write $|p|$ for the *length* of p, i.e. the number of positions in the sequence p. The i^{th} element of the sequence a is denoted as $(p)_{i-1}$. We write $p \frown q$ to denote the concatenation of the sequences p and q. In the following we make use of the lexicographic order $>^*_{\text{lex}}$ on $\mathbb{N}^*$. To simplify notation we write $>^{\text{lex}}$ instead of $>^*_{\text{lex}}$. Due to Lemma 2.2 we obtain that $\mathsf{otype}(>^{\text{lex}}) = \omega^\omega$, moreover in [21] it is shown that $\mathsf{otype}(\succ_{\mathsf{kbo}}) = \omega^\omega$ for finite signatures.

We generalise to infinite signature. Let $\succ$ be a precedence with $\mathsf{otype}(\succ) \leqslant \omega$ and let $\succ_{\mathsf{kbo}}$ be induced by $\succ$. Then $\mathsf{otype}(>^{\text{lex}}) = \mathsf{otype}(\succ_{\mathsf{kbo}}) = \omega^\omega$, a fact we exploit in later sections.

In the sequel of this paper we restrict to precedences of order type ω. This is motivated by our desired to obtain a clear comparison to Lepper's results as well as the observation that in termination proofs via KBO on infinitary signatures it is often possible to guarantee that $\mathsf{otype}(\succ) = \omega$, cf. Section 8. Furthermore we restrict our attention to signatures $\mathcal{F}$ with bounded arities (as in Example 3.1). The maximal arity of $\mathcal{F}$ is denoted as $\mathsf{ar}(\mathcal{F})$. For the general case of a precedence $\succ$ with $\mathsf{otype}(\succ) =: \alpha \geqslant \omega$ we can apply Lemma 2.2 and the pattern of the below presented proof to show that $\mathsf{otype}(\succ_{\mathsf{kbo}}) = \alpha^{\omega}$, if $\succ_{\mathsf{kbo}}$ is induced by $\succ$. However, our subsequent analysis of the derivation height of TRS compatible with KBO cannot be as easily generalised.

Definition 4.1. Let the signature $\mathcal{F}$ and a weight function (w, w_0) for $\mathcal{F}$ be fixed. We define an embedding $\mathsf{tw}\colon \mathcal{T}(\mathcal{F},\mathcal{V}) \to \mathbb{N}^*$. Set $b := \max\{\mathsf{ar}(\mathcal{F}), 3\} + 1$.

$$\mathsf{tw}(t) := \begin{cases} (w_0, a, 0) \frown 0^{\ell} & \text{if } t = \mathsf{i}^a x,\ x \in \mathcal{V}\ , \\ (\mathsf{w}(t), a, \mathsf{rk}(g)) \frown \mathsf{tw}(t_1) \frown \cdots \frown \mathsf{tw}(t_n) \frown 0^{\ell} & \text{if } t = \mathsf{i}^a g(t_1, \dots, t_n)\ . \end{cases}$$

The number ℓ is set suitably, so that $|\mathsf{tw}(t)| = b^{\mathsf{w}(t)+1}$. For readability of the definition we use the triple notation $(weight, a, rank)$ for $weight \frown a \frown rank$.

The mapping tw flattens a term t by transforming it into a concatenation of triples. Each triple holds the weight of the considered subterm r, the number of leading special symbols and the rank of the first non-special function symbol of r. In this way all the necessary information to compare two terms via $\succ_{\mathsf{kbo}}$ is expressed as a very simple data structure: a list of natural numbers. We remark that for signatures of order type $> \omega$ the rank r would not need to be a natural number.

Lemma 4.1. *The mapping* tw *embeds* $\succ_{\mathsf{kbo}}$ *into* $>^{\mathrm{lex}}$*: If* $s \succ_{\mathsf{kbo}} t$*, then* $\mathsf{tw}(s) >^{\mathrm{lex}} \mathsf{tw}(t)$*.*

Proof. The proof follows the pattern of the proof of Lemma 9 in [21].

Firstly, we make sure that the mapping tw is *well-defined*, i.e., we show that the length restriction can be met. We proceed by induction on t; let $t = \mathsf{i}^a t'$. We consider two cases (i) $t' \in \mathcal{V}$ or (ii) $t' = g(t_1, \dots, t_n)$. Suppose the former:

$$|(w_0, a, 0)| = 3 \leqslant b^{\mathsf{w}(t)+1}\ .$$

Now suppose case (ii): Let $j = \mathsf{rk}(g)$, we obtain

$$\begin{aligned} |(\mathsf{w}(t), a, j) \frown \mathsf{tw}(t_1) \frown \cdots \frown \mathsf{tw}(t_n)| &= 3 + b^{\mathsf{w}(t_1)+1} + \cdots + b^{\mathsf{w}(t_n)+1} \\ &\leqslant 3 + n \cdot b^{\mathsf{w}(t)} \leqslant b^{\mathsf{w}(t)+1}\ . \end{aligned}$$

Secondly, we show the following, slight generalisation of the lemma:

$$s \succ_{\mathsf{kbo}} t \wedge |\mathsf{tw}(s) \frown r| = |\mathsf{tw}(t) \frown r'| \quad \text{implies} \quad \mathsf{tw}(s) \frown r >^{\mathrm{lex}} \mathsf{tw}(t) \frown r'\ . \tag{1}$$

To prove (1) we proceed by induction on $s \succ_{\mathsf{kbo}} t$. Set $p = \mathsf{tw}(s) \frown r$, $q = \mathsf{tw}(t) \frown r'$.

Case $\mathsf{w}(s) > \mathsf{w}(t)$**:** By definition of the mapping tw, we have in this case that $(\mathsf{tw}(s))_0 > (\mathsf{tw}(t))_0$. Thus $p >^{\mathrm{lex}} q$ follows.

Case $\mathsf{w}(s) = \mathsf{w}(t)$**:** We consider the sub-case where $s = \mathsf{i}^a g(s_1, \dots, s_n)$ and $t = \mathsf{i}^a g(t_1, \dots, t_n)$ and there exists $i \in \{1, \dots, n\}$ such that $s_1 = t_1, \dots, s_{i-1} = t_{i-1}$, and $s_i \succ_{\mathsf{kbo}} t_i$. (The other cases are treated as in the case above.) The induction hypothesis expresses that if $|\mathsf{tw}(s_i) \frown v| = |\mathsf{tw}(t_i) \frown v'|$, then $\mathsf{tw}(s_i) \frown v >^{\mathrm{lex}} \mathsf{tw}(t_i) \frown v'$. For $j = \mathsf{rk}(g)$, we obtain

$$\begin{aligned} p &= \overbrace{(\mathsf{w}(s), a, j) \frown \mathsf{tw}(s_1) \frown \cdots \frown \mathsf{tw}(s_{i-1})}^{w} \frown \mathsf{tw}(s_i) \frown \cdots \frown \mathsf{tw}(s_n) \frown r \ , \\ q &= \underbrace{(\mathsf{w}(s), a, j) \frown \mathsf{tw}(s_1) \frown \cdots \frown \mathsf{tw}(s_{i-1})}_{w} \frown \mathsf{tw}(t_i) \frown \cdots \frown \mathsf{tw}(t_n) \frown r' \ . \end{aligned}$$

Due to $|p| = |q|$, we conclude

$$|\mathsf{tw}(s_i) \frown \cdots \frown \mathsf{tw}(s_n) \frown r| = |\mathsf{tw}(t_i) \frown \cdots \frown \mathsf{tw}(t_n) \frown r'| \ .$$

Hence induction hypothesis is applicable and we obtain

$$\mathsf{tw}(s_i) \frown \cdots \frown \mathsf{tw}(s_n) \frown r >^{\mathrm{lex}} \mathsf{tw}(t_i) \frown \cdots \frown \mathsf{tw}(t_n) \frown r' \ ,$$

which yields $p >^{\mathrm{lex}} q$. This completes the proof of (1).

Finally, to establish the lemma, we assume $s \succ_{\mathsf{kbo}} t$. By definition either $\mathsf{w}(s) > \mathsf{w}(t)$ or $\mathsf{w}(s) = \mathsf{w}(t)$. In the latter case $\mathsf{tw}(s) >^{\mathrm{lex}} \mathsf{tw}(t)$ follows by (1). While in the former $\mathsf{tw}(s) >^{\mathrm{lex}} \mathsf{tw}(t)$ follows as $\mathsf{w}(s) > \mathsf{w}(t)$ implies $|\mathsf{tw}(s)| > |\mathsf{tw}(t)|$. ☐

From the lemma we conclude the existence of an embedding from $\succ_{\mathsf{kbo}}$ into $>^{\mathrm{lex}}$. Thus $\mathsf{otype}(\succ_{\mathsf{kbo}}) \leqslant \omega^\omega$ over infinitary signatures, if the above conditions are fulfilled. In order to show $\mathsf{otype}(\succ_{\mathsf{kbo}}) = \omega^\omega$ we can rely on a result by Lepper [21, Lemma 8] that yields that the order type of $\succ_{\mathsf{kbo}}$ is at least ω^ω, even if restricted to ground terms.

5 Derivation Height of Knuth-Bendix Orders

Let $\mathcal{R}$ be a TRS and $\succ_{\mathsf{kbo}}$ a KBO induced by a precedence $\succ$ such that $\mathsf{otype}(\succ) = \omega$. Furthermore suppose $\succ_{\mathsf{kbo}}$ is compatible with $\mathcal{R}$; if not stated otherwise $\mathcal{R}$ and $\succ_{\mathsf{kbo}}$ are fixed for the remainder of the section. In this section we establish an upper bound on the derivation height with respect to $\mathcal{R}$.

We introduce a couple of *measure functions* for term and sequence complexities, respectively. The first measure $\mathsf{sp}\colon \mathcal{T}(\mathcal{F}, \mathcal{V}) \to \mathbb{N}$ bounds the maximal nesting of special symbols in the term:

$$\mathsf{sp}(t) := \begin{cases} a & \text{if } t = \mathsf{i}^a x,\ x \text{ a variable} \ , \\ \max(\{a\} \cup \{\mathsf{sp}(t_j) \mid j \in \{1, \dots, n\}\}) & \text{if } t = \mathsf{i}^a g(t_1, \dots, t_n) \ . \end{cases}$$

The second and third measure $\mathsf{rkt}\colon \mathcal{T}(\mathcal{F},\mathcal{V}) \to \mathbb{N}$ and $\mathsf{mrk}\colon \mathcal{T}(\mathcal{F},\mathcal{V}) \to \mathbb{N}$ collect information on the ranks of non-special function symbols occurring:

$$\mathsf{rkt}(t) := \begin{cases} 0 & \text{if } t = \mathsf{i}^a x,\ x \in \mathcal{V}\ , \\ j & \text{if } t = \mathsf{i}^a g(t_1,\dots,t_n) \text{ and } \mathsf{rk}(g) = j\ , \end{cases}$$

$$\mathsf{mrk}(t) := \begin{cases} 0 & \text{if } t = \mathsf{i}^a x,\ x \in \mathcal{V}\ , \\ \max(\{j\} \cup \{\mathsf{mrk}(t_i) \mid i \in \{1,\dots,n\}\}) & \text{if } t = \mathsf{i}^a g(t_1,\dots,t_n),\ \mathsf{rk}(g) = j\ . \end{cases}$$

The fourth measure $\mathsf{max}\colon \mathbb{N}^* \to \mathbb{N}$ considers sequences p and bounds the maximal number occurring in p:

$$\mathsf{max}(p) := \max\{(p)_i \mid i \in \{0,\dots,|p|-1\}\}\ .$$

It is immediate from the definitions that for any term t: $\mathsf{sp}(t), \mathsf{rkt}(t), \mathsf{mrk}(t) \leqslant \mathsf{max}(\mathsf{tw}(t))$.

Lemma 5.1. *If $r \trianglelefteq t$, then $\mathsf{max}(\mathsf{tw}(t)) \geqslant \mathsf{max}(\mathsf{tw}(r))$.*

We informally argue for the correctness of the lemma. Suppose r is a subterm of t. Then clearly $\mathsf{w}(r) \leqslant \mathsf{w}(t)$. The maximal occurring nesting of special symbols in r is smaller (or equal) than in t. And the maximal rank of a symbol in r is smaller (or equal) than in t. The mapping tw transforms r to a sequence whose coefficients are less than $\mathsf{w}(t)$, less than the maximal nesting of special symbols and less than the maximal rank of non-special function symbol in r . Hence $\mathsf{max}(\mathsf{tw}(t)) \geqslant \mathsf{max}(\mathsf{tw}(r))$ holds.

Lemma 5.2. *If $p = \mathsf{tw}(t)$ and $q = \mathsf{tw}(\mathsf{i}^a t)$, then $\mathsf{max}(p) + a \geqslant \mathsf{max}(q)$.*

Proof. The proof of the lemma proceeds by a case distinction on t. □

Lemma 5.3. *We write $m \dot{-} n$ to denote $\max\{m-n, 0\}$. Assume $s \succ_{\mathsf{kbo}} t$ with $\mathsf{sp}(t) \leqslant K$ and $(\mathsf{mrk}(t) \dot{-} \mathsf{rkt}(s)) \leqslant K$. Let σ be a substitution and set $p = \mathsf{tw}(s\sigma)$, $q = \mathsf{tw}(t\sigma)$. Then $p >^{\mathrm{lex}} q$ and $\mathsf{max}(p) + K \geqslant \mathsf{max}(q)$.*

Proof. It suffices to show $\mathsf{max}(p) + K \geqslant \mathsf{max}(q)$ as $p >^{\mathrm{lex}} q$ follows from Lemma 4.1. We proceed by induction on t; let $t = \mathsf{i}^a t'$.

Case t' is a variable: Set $t' = x$. We consider two sub-cases: Either (i) $x\sigma = \mathsf{i}^b y$, $y \in \mathcal{V}$ or (ii) $x\sigma = \mathsf{i}^b g(u_1,\dots,u_m)$. It suffices to consider sub-case (ii), as sub-case (i) is treated in a similar way. From $s \succ_{\mathsf{kbo}} t$, we know that for all $y \in \mathcal{V}$, $|s|_y \geqslant |t|_y$, hence $x \in \mathcal{V}\mathsf{ar}(s)$ and $x\sigma \trianglelefteq s\sigma$. Let $l := \mathsf{rk}(g)$; by Lemma 5.1 we conclude $\mathsf{max}(\mathsf{tw}(x\sigma)) \leqslant \mathsf{max}(p)$. That is, $b, l, \mathsf{max}(\mathsf{tw}(u_1)), \dots, \mathsf{max}(\mathsf{tw}(u_m)) \leqslant \mathsf{max}(p)$. We obtain

$$\begin{aligned} \mathsf{max}(q) &= \max(\{\mathsf{w}(t\sigma), a+b, l\} \cup \{\mathsf{max}(\mathsf{tw}(u_j)) \mid i \in \{1,\dots,m\}\}) \\ &\leqslant \max(\{\mathsf{w}(s\sigma), \mathsf{sp}(t) + \mathsf{max}(p), \mathsf{max}(p)\} \cup \{\mathsf{max}(p)\}) \\ &\leqslant \max(\{\mathsf{w}(s\sigma), \mathsf{max}(p) + K\} \cup \{\mathsf{max}(p)\}) = \mathsf{max}(p) + K\ . \end{aligned}$$

Case $t' = g(t_1, \dots, t_n)$**:** Let $j = \mathsf{rk}(g)$. By Definition 3.1 we obtain $s \succ_{\mathsf{kbo}} t_i$. Moreover $\mathsf{sp}(t_i) \leqslant \mathsf{sp}(t) \leqslant K$ and $\mathsf{mrk}(t_i) \leqslant \mathsf{mrk}(t)$. Hence for all i: $\mathsf{sp}(t_i) \leqslant K$ and $(\mathsf{mrk}(t_i) \dot{-} \mathsf{rkt}(s)) \leqslant K$ holds. Thus induction hypothesis is applicable: For all i: $\mathsf{max}(\mathsf{tw}(t_i\sigma)) \leqslant \mathsf{max}(p) + K$. By using the assumption $(\mathsf{mrk}(t) \dot{-} \mathsf{rkt}(s)) \leqslant K$ we obtain:

$$\begin{aligned}
\mathsf{max}(q) &= \max(\{\mathsf{w}(t\sigma), a, j\} \cup \{\mathsf{max}(\mathsf{tw}(t_i\sigma)) \mid i \in \{1, \dots, n\}\}) \\
&\leqslant \max(\{\mathsf{w}(t\sigma), \mathsf{sp}(t), \mathsf{rkt}(s) + K\} \cup \{\mathsf{max}(p) + K\}) \\
&\leqslant \max(\{\mathsf{w}(s\sigma), \mathsf{sp}(t), \mathsf{rkt}(s\sigma) + K\} \cup \{\mathsf{max}(p) + K\}) \\
&\leqslant \max(\{\mathsf{w}(s\sigma), K, \mathsf{max}(p) + K\} \cup \{\mathsf{max}(p) + K\}) = \mathsf{max}(p) + K \ .
\end{aligned}$$

□

In the following, we assume that the set

$$M := \{\mathsf{sp}(r) \mid l \to r \in \mathcal{R}\} \cup \{(\mathsf{mrk}(r) \dot{-} \mathsf{rkt}(l)) \mid l \to r \in \mathcal{R}\} \tag{2}$$

is finite. We set $K := \max M$ and let K be fixed for the remainder.

Example 5.1 (Example 3.1 continued)**.** With respect to $\mathcal{R}_2$, we have

$$M = \{(\mathsf{mrk}(r) \dot{-} \mathsf{rkt}(l)) \mid l \to r \in \mathcal{R}_2\} \ .$$

Note that the signature of $\mathcal{R}_2$ doesn't contain a special symbol. M is finite and it is easy to see that $\max M = 1$.

Exemplary, we consider the rule schemata $\mathsf{f}_{n+1}(\mathsf{h}(x)) \to \mathsf{f}_n(\mathsf{i}(x))$. Observe that the rank of i equals 4, the rank of h is 3, and the rank of f_n is given by $n + 5$. Hence $\mathsf{mrk}(f_n(i(x))) = n + 5$ and $\mathsf{rkt}(f_{n+1}(h(x))) = n + 6$. Clearly $(n + 5 \dot{-} n + 6) \leqslant 1$.

Lemma 5.4. *If $s \to_{\mathcal{R}} t$, $p = \mathsf{tw}(s)$, $q = \mathsf{tw}(t)$, then $p >^{\mathrm{lex}} q$ and $u(\mathsf{max}(p), K) \geqslant \mathsf{max}(q)$, where u denotes a monotone polynomial such that $u(m, n) \geqslant 2m + n$.*

Proof. By definition of the rewrite relation there exists a context C, a substitution σ and a rule $l \to r \in R$ such that $s = C[l\sigma]$ and $t = C[r\sigma]$. We prove $\mathsf{max}(q) \leqslant u(\mathsf{max}(p), K)$ by induction on C. Note that C can only have the form (i) $C = \mathsf{i}^a[\Box]$ or (ii) $C = \mathsf{i}^a g(u_1, \dots, C'[\Box], \dots, u_n)$ for some $a \in \mathbb{N}$.

Case $C = \mathsf{i}^a[\Box]$**:** By Lemma 5.3 we see $\mathsf{max}(\mathsf{tw}(r\sigma)) \leqslant \mathsf{max}(\mathsf{tw}(l\sigma)) + K$. Employing in addition Lemma 5.2 and Lemma 5.1, we obtain:

$$\begin{aligned}
\mathsf{max}(q) &= \mathsf{max}(\mathsf{tw}(\mathsf{i}^a r\sigma)) \leqslant \mathsf{max}(\mathsf{tw}(r\sigma)) + a \\
&\leqslant \mathsf{max}(\mathsf{tw}(l\sigma)) + K + a \\
&\leqslant \mathsf{max}(p) + K + \mathsf{max}(p) \leqslant u(\mathsf{max}(p), K) \ .
\end{aligned}$$

Case $C = \mathsf{i}^a g(u_1, \ldots, C'[\Box], \ldots, u_n)$**:** As $C'[l\sigma] \to_{\mathcal{R}} C'[r\sigma]$, induction hypothesis is applicable: Let $p' = \mathsf{tw}(C'[l\sigma])$, $q' = \mathsf{tw}(C'[r\sigma])$. Then $\mathsf{max}(q') \leqslant u(\mathsf{max}(p'), K)$. For $\mathsf{rk}(g) = l$, we obtain by application of induction hypothesis and Lemma 5.1:

$$\begin{aligned}\mathsf{max}(q) &= \max(\{\mathsf{w}(t), a, l\} \cup \{\mathsf{max}(\mathsf{tw}(u_1)), \ldots, \mathsf{max}(q'), \ldots, \mathsf{max}(\mathsf{tw}(u_n))\}) \\ &\leqslant \max(\{\mathsf{w}(s), a, l\} \cup \{\mathsf{max}(\mathsf{tw}(u_1)), \ldots, u(\mathsf{max}(p'), K), \ldots, \mathsf{max}(\mathsf{tw}(u_n))\}) \\ &\leqslant \max(\{\mathsf{w}(s), a, l\} \cup \{\mathsf{max}(p), u(\mathsf{max}(p), K)\}) = u(\mathsf{max}(p), K) \ .\end{aligned}$$

□

Adapting an idea of Buchholz [6], we define a family of *approximations* $(>^{\text{lex}}_n)_{n \geqslant 0}$ of the proper order $>^{\text{lex}}$ as follows: $p >^{\text{lex}}_n q$ if

$$p >^{\text{lex}} q \text{ and } u(\mathsf{max}(p), n) \geqslant \mathsf{max}(q) \ ,$$

where $u(m, n) \geqslant 2m + n$. We remark that the approximation $>^{\text{lex}}_n$ need not be a proper order. The following example shows that transitivity may fail.

Example 5.2. Consider the following sequences of length 3: $\mathsf{100}$, $\mathsf{020}$, and $\mathsf{004}$. Clearly we have $\mathsf{100} >^{\text{lex}} \mathsf{020} >^{\text{lex}} \mathsf{004}$ and of course also $\mathsf{100} >^{\text{lex}} \mathsf{004}$. We consider the approximation $>^{\text{lex}}_0$ and set $u(m, n) := 2m + n$ so that we obtain:

$$\mathsf{100} >^{\text{lex}}_0 \mathsf{020} >^{\text{lex}}_0 \mathsf{004} \ .$$

Here the first inequality follows as $\mathsf{max}(\mathsf{020}) = 2 = u(1, 0) = u(\mathsf{max}(\mathsf{100}), 0)$; similarly the second inequality follows as $\mathsf{max}(\mathsf{020}) = 4 = u(\mathsf{max}(\mathsf{020}, 0))$. However $\mathsf{100} \not>^{\text{lex}}_0 \mathsf{004}$ as $\mathsf{max}(\mathsf{020}) \not\leqslant u(\mathsf{max}(\mathsf{100}), 0)$.

Based on the above definition Lemma 5.3 can be succinctly expressed as follows, for K as above.

Lemma 5.5. *If $s \to_{\mathcal{R}} t$, then $\mathsf{tw}(s) >^{\text{lex}}_K \mathsf{tw}(t)$.*

Definition 5.1. In the spirit of the definition of derivation height, we define a family of functions $\mathsf{Ah}_n \colon \mathbb{N} \to \mathbb{N}$:

$$\mathsf{Ah}_n(p) := \max\{k \mid \exists p_0 \cdots p_k \ p = p_0 >^{\text{lex}}_n p_1 >^{\text{lex}}_n \cdots >^{\text{lex}}_n p_k\} \ .$$

Theorem 5.1. *Let $\mathcal{F}$ be a signature with bounded arities and let $\mathcal{R}$ be a TRS over $\mathcal{F}$. Suppose $\mathcal{R}$ is compatible with KBO based on a precedence of order type ω such that the set (2) is finite. Set $K := \max M$. Then $\mathsf{dheight}_{\mathcal{R}}(t) \leqslant \mathsf{Ah}_K(\mathsf{tw}(t))$.*

Proof. By assumption there exists an instance of KBO compatible with $\mathcal{R}$, that is, $\mathcal{R} \subseteq \succ_{\mathsf{kbo}}$. Thus any derivation D over $\mathcal{R}$ with start term t is finite:

$$D \colon t = t_0 \to t_1 \to \cdots \to t_k \ .$$

Due to Lemma 5.5 derivation D yields the following descending sequence of the binary relation $>^{\text{lex}}_K$:

$$\mathsf{tw}(t) = \mathsf{tw}(t_0) >^{\text{lex}}_K t_1 >^{\text{lex}}_K \cdots >^{\text{lex}}_K \mathsf{tw}(t_k) \ .$$

By Definition 5.1 the length of this descending sequence is bounded by $\mathsf{Ah}_K(\mathsf{tw}(t))$ as claimed. □

In the next section we show that Ah_n is bounded by the Ackermann function. Thus providing the sought upper bound on the derivation height of $\mathcal{R}$.

6 Bounding the Growth of Ah_n

Instead of directly relating the functions Ah_n to the Ackermann function, we make use of the fast-growing *Hardy* functions, cf. [33]. The Hardy functions form a hierarchy of unary functions $\mathsf{H}_\alpha \colon \mathbb{N} \to \mathbb{N}$ indexed by ordinals. We will only be interested in a small part of this hierarchy, namely in the set of functions $\{\mathsf{H}_\alpha \mid \alpha < \omega^\omega\}$.

Definition 6.1. We define the embedding $\mathsf{o} \colon \mathbb{N}^* \to \omega^\omega$ as follows:

$$\mathsf{o}(p) := \omega^{\ell-1}(p)_0 + \dots \omega(p)_{\ell-2} + (p)_{\ell-1} \ ,$$

where $\ell = |p|$.

The next lemma follows directly from the definitions.

Lemma 6.1. *If $p >^{\mathrm{lex}} q$, then $\mathsf{o}(p) > \mathsf{o}(q)$.*

We associate with every $\alpha < \omega^\omega$ in *CNF* an ordinal α_n, where $n \in \mathbb{N}$. The sequence $(\alpha_n)_n$ is called *fundamental sequence* of α. (For the connection between rewriting and fundamental sequences see e.g. [31].)

$$\alpha_n := \begin{cases} 0 & \text{if } \alpha = 0 \ , \\ \beta & \text{if } \alpha = \beta + 1 \ , \\ \beta + \omega^{\gamma+1} \cdot (k-1) + \omega^\gamma \cdot (n+1) & \text{if } \alpha = \beta + \omega^{\gamma+1} \cdot k \ . \end{cases}$$

Based on the definition of α_n, we define $\mathsf{H}_\alpha \colon \mathbb{N} \to \mathbb{N}$, for $\alpha < \omega^\omega$ by transfinite induction on α:

$$\mathsf{H}_0(n) := n \qquad \mathsf{H}_\alpha(n) := \mathsf{H}_{\alpha_n}(n+1) \ .$$

Let $>_{(n)}$ denote the transitive closure of $(.)_n$, i.e. $\alpha >_{(n)} \beta$ iff $\alpha_n >_{(n)} \beta$ or $\alpha_n = \beta$. Suppose $\alpha, \beta < \omega^\omega$. Let $\alpha = \omega^{\alpha_1} n_1 + \dots \omega^{\alpha_k} n_k$ and $\beta = \omega^{\beta_1} m_1 + \dots \omega^{\beta_l} m_l$. Recall that any ordinal $\alpha \neq 0$ can be uniquely written in CNF. Hence we can assume that $\alpha_1 > \cdots > \alpha_k$ and $\beta_1 > \cdots > \beta_l$. Furthermore by our assumption that $\alpha, \beta < \omega^\omega$, we have $\alpha_i, \beta_j \in \mathbb{N}$. We write $\mathsf{NF}(\alpha, \beta)$ if $\alpha_k \geqslant \beta_1$.

Before we proceed in our estimation of the functions Ah_n, we state some simple facts that help us to calculate with the function H_α.

Lemma 6.2.

1. *If $\alpha >_{(n)} \beta$, then $\alpha >_{(n+1)} \beta + 1$ or $\alpha = \beta + 1$.*
2. *If $\alpha >_{(n)} \beta$ and $m \geqslant n$, then $\mathsf{H}_\alpha(m) > \mathsf{H}_\beta(m)$.*
3. *If $n > m$, then $\mathsf{H}_\alpha(n) > \mathsf{H}_\alpha(m)$.*
4. *If $\mathsf{NF}(\alpha, \beta)$, then $\mathsf{H}_{\alpha+\beta}(n) = \mathsf{H}_\alpha \circ \mathsf{H}_\beta(n)$; $\circ$ denotes function composition.*

We relate the Hardy functions with the Ackermann function. The stated upper bound is a gross one, but a more careful estimation is not necessary here.

Lemma 6.3. *For $n \geqslant 1$: $\mathsf{H}_{\omega^n}(m) \leqslant \mathsf{Ack}(2n, m)$.*

Proof. We recall the definition of the Ackermann function:

$$\begin{aligned}\mathsf{Ack}(0, m) &= m + 1 \\ \mathsf{Ack}(n + 1, 0) &= \mathsf{Ack}(n, 1) \\ \mathsf{Ack}(n + 1, m + 1) &= \mathsf{Ack}(n, \mathsf{Ack}(n + 1, m))\end{aligned}$$

In the following we sometimes denote the Ackermann function as a unary function, indexed by its first argument: $\mathsf{Ack}(n, m) = \mathsf{Ack}_n(m)$. To prove the lemma, we proceed by induction on the lexicographic comparison of n and m. We only present the case, where n and m are greater than 0. As preparation note that $m + 1 \leqslant \mathsf{H}_{\omega^n}(m)$ holds for any n and $\mathsf{Ack}^2_n(m + 1) \leqslant \mathsf{Ack}_{n+1}(m + 1)$ holds for any n, m.

$$\begin{aligned}\mathsf{H}_{\omega^{n+1}}(m + 1) &= \mathsf{H}_{\omega^n(m+2)}(m + 2) \\ &\leqslant \mathsf{H}_{\omega^n(m+2)+\omega^n}(m + 1) && \text{Lemma 6.2(3,4)} \\ &= \mathsf{H}^2_{\omega^n}\mathsf{H}_{\omega^n(m+1)}(m + 1) && \text{Lemma 6.2(4)} \\ &= \mathsf{H}^2_{\omega^n}\mathsf{H}_{\omega^{n+1}}(m) \\ &\leqslant \mathsf{Ack}^2_{2n}\mathsf{Ack}_{2(n+1)}(m) && \text{induction hypothesis} \\ &\leqslant \mathsf{Ack}_{2n+1}\mathsf{Ack}_{2(n+1)}(m) \\ &= \mathsf{Ack}(2(n + 1), m + 1)\ .\end{aligned}$$

□

Lemma 6.4. *Assume $u(m, n) \leqslant 2m + n$ and set $\ell = |p|$. For all $n \in \mathbb{N}$:*

$$\mathsf{Ah}_n(p) \leqslant \mathsf{H}_{\omega^2\cdot\mathsf{o}(p)}(u(\mathsf{max}(p), n) + 1) < \mathsf{H}_{\omega^{4+\ell}}(\mathsf{max}(p) + n)\ . \tag{3}$$

Proof. To prove the first half of (3), we make use of the following fact:

$$p >^{\text{lex}} q \wedge n \geqslant \mathsf{max}(q) \quad \text{implies} \quad \mathsf{o}(p) >_{(n)} \mathsf{o}(q)\ . \tag{4}$$

To prove (4), one proceeds by induction on $>^{\text{lex}}$ and uses that the embedding $\mathsf{o}\colon \mathbb{N}^* \to \omega^\omega$ is an order-isomorphism. We omit the details.

By definition, we have $\mathsf{Ah}_n(p) = \max\{\mathsf{Ah}_n(q) + 1 \mid p >^{\text{lex}}_n q\}$. Hence it suffices to prove

$$p >^{\text{lex}} q \wedge u(\mathsf{max}(p), n) \geqslant \mathsf{max}(q) \quad \text{implies} \quad \mathsf{Ah}_n(q) < \mathsf{H}_{\omega^2\cdot\mathsf{o}(p)}(u(\mathsf{max}(p), n) + 1) \tag{5}$$

We fix p fulfilling the assumptions in (5); let $\alpha = \mathsf{o}(p)$, $\beta = \mathsf{o}(q)$, $v = u(\mathsf{max}(p), n)$. We use (4) to obtain $\alpha >_{(v)} \beta$. We proceed by induction on $>^{\text{lex}}$.

Case $\alpha_v = \beta$**:** As $p >^{\mathrm{lex}} q$, we conclude $\mathsf{Ah}_n(q) \leqslant \mathsf{H}_{\omega^2\cdot\mathsf{o}(q)}(u(\mathsf{max}(q), n) + 1)$ due to induction hypothesis. It is not difficult to see that for any $p \in \mathbb{N}^*$ and $n \in \mathbb{N}$, $4\,\mathsf{max}(p)+2n+1 \leqslant \mathsf{H}_{\omega^2}(u(\mathsf{max}(p), n))$. In sum, we obtain:

$$
\begin{aligned}
\mathsf{Ah}_n(q) &\leqslant \mathsf{H}_{\omega^2\cdot\mathsf{o}(q)}(u(\mathsf{max}(q), n) + 1) && \\
&\leqslant \mathsf{H}_{\omega^2\cdot\alpha_v}(u(u(\mathsf{max}(p), n), n) + 1) && \mathsf{max}(q) \leqslant u(\mathsf{max}(p), n) \\
&\leqslant \mathsf{H}_{\omega^2\cdot\alpha_v}(4\,\mathsf{max}(p) + 2n + 1) && \text{Definition of } u \\
&\leqslant \mathsf{H}_{\omega^2\cdot\alpha_v}\mathsf{H}_{\omega^2}(u(\mathsf{max}(p), n)) && \\
&= \mathsf{H}_{\omega^2\cdot(\alpha_v+1)}(u(\mathsf{max}(p), n)) && \text{Lemma 6.2(4) and definition} \\
& && \text{of ordinal multiplication} \\
&< \mathsf{H}_{\omega^2\cdot(\alpha_v+1)}(u(\mathsf{max}(p), n) + 1) && \text{Lemma 6.2(3)} \\
&\leqslant \mathsf{H}_{\omega^2\cdot\alpha}(u(\mathsf{max}(p), n) + 1) && \text{Lemma 6.2(2)}
\end{aligned}
$$

The application of Lemma 6.2(2) in the last step is feasible as by definition $\alpha >_{(v)} \alpha_v$. An application of Lemma 6.2(1) yields $\alpha_v + 1 \leq_{(v+1)} \alpha$. From which we deduce $\omega^2 \cdot (\alpha_v + 1) \leq_{(v+1)} \omega^2 \cdot \alpha$.

Case $\alpha_v >_{(v)} \beta$**:** In this case the proof follows the pattern of the above proof, but an additional application of Lemma 6.2(4) is required. This completes the proof of(5).

To prove the second part of (3), we proceed as follows: The fact that $\omega^\ell > \mathsf{o}(p)$ is immediate from the definitions. Induction on p reveals that even $\omega^\ell >_{(\mathsf{max}(p))} \mathsf{o}(p)$ holds. Thus in conjunction with the above, we obtain:

$$
\begin{aligned}
\mathsf{Ah}_n(p) &\leqslant \mathsf{H}_{\omega^2\cdot\mathsf{o}(p)}(u(\mathsf{max}(p), n) + 1) \\
&\leqslant \mathsf{H}_{\omega^{2+\ell}}(u(\mathsf{max}(p), n) + 1) \\
&\leqslant \mathsf{H}_{\omega^{4+\ell}}(\mathsf{max}(p) + n)\ .
\end{aligned}
$$

The last step follows as $2\,\mathsf{max}(p) + n + 1 \leqslant \mathsf{H}_{\omega^2}(\mathsf{max}(p) + n)$ and $\ell < \omega$. □

As a consequence of Lemma 6.3 and 6.4, we obtain the following theorem.

Theorem 6.1. *For all* $n \geqslant 1$*:* $\mathsf{Ah}_n(p) \leqslant \mathsf{Ack}(2|p| + 8, \mathsf{max}(p) + n)$.

7 Derivation Height of Rewrite Systems Compatible with Knuth-Bendix Orders

Based on Theorem 5.1 and 6.1 we obtain that the derivation height of $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is bounded in the Ackermann function.

Theorem 7.1. *Let* $\mathcal{R}$ *be a TRS over a signature* $\mathcal{F}$ *of bounded arity. Suppose* $\mathcal{R}$ *is compatible with* $\succ_{\mathsf{kbo}}$ *induced by a precedence of order type* ω*. Furthermore, assume the set* $M := \{\mathsf{sp}(r) \mid l \to r \in \mathcal{R}\} \cup \{(\mathsf{mrk}(r) \dot{-} \mathsf{rkt}(l)) \mid l \to r \in \mathcal{R}\}$ *is finite; set* $K := \max M$*. Then*

$$\mathsf{dheight}(t, \to_{\mathcal{R}}) \leqslant \mathsf{Ack}(2|\mathsf{tw}(t)| + \mathsf{max}(\mathsf{tw}(t)) + K + 8, 0)\ .$$

More succinctly we have $\mathsf{dheight}(t, \to_{\mathcal{R}}) = \mathsf{Ack}(O(\max\{|\mathsf{tw}(t)|, \mathsf{max}(\mathsf{tw}(t)), K\}), 0)$.

Proof. We set $u(m, n) = 2m + n$ and keep the definition of u fixed for the sequel of the section; furthermore let $p = \mathsf{tw}(t)$.

Due to Theorem 5.1 we conclude that $\mathsf{dheight}(t, \to_{\mathcal{R}}) \leqslant \mathsf{Ah}_K(p)$. It is easy to see that $\mathsf{Ack}(n, m) \leqslant \mathsf{Ack}(n + m, 0)$. Using this fact and Theorem 6.1 we obtain:

$$\mathsf{Ah}_K(p) \leqslant \mathsf{Ack}(2|p| + 8, \mathsf{max}(p) + K) \leqslant \mathsf{Ack}(2|p| + \mathsf{max}(p) + K + 8, 0) .$$

Thus the theorem follows. ☐

For fixed $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ we can bound the argument of the Ackermann function in the above theorem in terms of the size of t. We define

$$r_{\max} := \mathsf{mrk}(t) \qquad\qquad w_{\max} := \max(\{\mathsf{w}(f) \mid f \in \mathcal{F}(t)\} \cup \{w_0\}) .$$

Lemma 7.1. *For t be a term, let $r_{\max}$, $w_{\max}$ be as above. Let $b := \max\{\mathsf{ar}(\mathcal{F}), 3\} + 1$, and set $n := \mathsf{sz}(t)$. Then $\mathsf{w}(t) \leqslant w_{\max} \cdot n$ and $\mathsf{sp}(t) \leqslant n$. Hence $|\mathsf{tw}(t)| \leqslant b^{w_{\max} \cdot n+1}$ and $\mathsf{max}(\mathsf{tw}(t)) \leqslant w_{\max} \cdot n + r_{\max}$.*

Proof. The proof proceeds by induction on t. ☐

Corollary 7.1. *Let $\mathcal{R}$ be a TRS that fulfils the properties in the theorem. Then there exists a constant c—depending on $\mathcal{R}$, $r_{\max}$, and $w_{\max}$ only— such that* $\mathsf{dheight}(t, \to_{\mathcal{R}}) \leqslant \mathsf{Ack}(c^n, 0)$*, whenever* $\mathsf{sz}(t) \leqslant n$. *More succinctly we have* $\mathsf{dheight}(t, \to_{\mathcal{R}}) = \mathsf{Ack}(2^{O(\mathsf{sz}(t))}, 0)$.

Proof. Combining Theorem 7.1 and Lemma 7.1 we obtain:

$$\begin{aligned}
\mathsf{dheight}(t, \to_{\mathcal{R}}) &\leqslant \mathsf{Ack}(2|\mathsf{tw}(t)| + \mathsf{max}(\mathsf{tw}(t)) + K + 8, 0) \\
&\leqslant \mathsf{Ack}(2 \cdot b^{w_{\max} \cdot n+1} + (w_{\max} \cdot n + r_{\max}) + K + 8, 0) \\
&\leqslant \mathsf{Ack}((\max\{b, K, 8\})^{w_{\max} \cdot n + r_{\max} + 2}, 0) \\
&\leqslant \mathsf{Ack}((\max\{b, K, 8\})^{(w_{\max} + r_{\max} + 2) \cdot n}, 0) \\
&= \mathsf{Ack}(\Big((\max\{b, K, 8\})^{(w_{\max} + r_{\max} + 2)}\Big)^n, 0) .
\end{aligned}$$

Setting $c := (\max\{b, K, 8\})^{(w_{\max} + r_{\max} + 2)}$ the corollary follows. ☐

Example 7.1 (Example 3.1 continued)**.** Let $t \in \mathcal{T}(\mathcal{F}_{\text{lab}}, \mathcal{V})$ be fixed and set $n := \mathsf{sz}(t)$. The arities of the function symbols in $\mathcal{F}_{\text{lab}}$ are bounded and in Example 5.1 we indicated that for $M = \{(\mathsf{mrk}(r) \dot{-} \mathsf{rkt}(l)) \mid l \to r \in \mathcal{R}_2\}$, we have $\max M = 1$. Corollary 7.1 is applicable to conclude that $\mathsf{dheight}(t, \to_{\mathcal{R}_2}) = \mathsf{Ack}(2^{O(n)}, 0)$.

Note that it is not straight-forward to apply Theorem 7.1 to classify the *derivational complexity* of $\mathcal{R}$, over infinite signature, compatible with KBO. This is clarified in the next example.

Example 7.2. Consider the TRS $\mathcal{R}_3$ consisting of all rules of the form:

$$\mathsf{f}_{i+1} \to \mathsf{f}_i \qquad \text{for all } i \geqslant 0 .$$

Setting $\mathsf{w}(\mathsf{f}_i) = 1$ and $\mathsf{f}_{i+1} \succ \mathsf{f}_i$ we obtain $\mathcal{R}_3 \subseteq \succ_{\mathsf{kbo}}$. Furthermore the arity of the function symbols is trivially bounded to 0 and for $M := \{(\mathsf{mrk}(r) \dot{-} \mathsf{rkt}(l)) \mid l \to r \in \mathcal{R}\}$, we have $\max M = 0$. Thus Corollary 7.1 is applicable, to conclude that for all f_i there exists a constant c_i such that $\mathsf{dheight}(\mathsf{f}_i, \to_{\mathcal{R}_2}) = \mathsf{Ack}(2^{c_1}, 0)$. On the other hand the derivational complexity function $\mathsf{dc}_{\mathcal{R}_3}$ is undefined as no natural number can bound the arbitrarily large derivation heights of size 1 terms.

Example 7.2 may be considered pathological and thus of limited significance. While we cannot conceptually improve this for KBOs, we show that it is not possible to bind the derivational complexity function $\mathsf{dc}_{\mathcal{R}_3}$ of *precedence terminating* TRSs. Here a TRS $\mathcal{R}$ is precedence terminating, if there exists a well-founded proper order $\succ$ such that for all $l \to r \in \mathcal{R}$ and any non-variable subterm u of r, $\mathsf{rt}(l) \succ \mathsf{rt}(u)$. Precedence termination implies termination. We make use of the following proposition by Middeldorp and Zantema.

Proposition 7.1 ([26]). *Every terminating $\mathcal{R}$ admits a well-founded quasi-model $(\mathcal{A}, >)$ with weakly monotone labelling ℓ such that $\mathcal{R}_{\mathrm{lab}} \cup \mathcal{D}ec(>)$ is precedence terminating.*

The main idea of the proof is that $(\mathcal{T}(\mathcal{F}, \mathcal{V}), \to_{\mathcal{R}})$ forms a quasi-model of $\mathcal{R}$, if $\mathcal{R}$ is terminating. Thus *self-labelling* of left- and right-hand sides of rules yields a precedence terminating TRS $\mathcal{R}_{\mathrm{lab}}$ with respect to the proper order $(\to_{\mathcal{R}} \cup \rhd)^+$.

Lemma 7.2. *There is no subrecursive class $\mathcal{C}$ of functions (over the naturals) such that for all precedence terminating TRS $\mathcal{R}$, $\mathsf{dc}_{\mathcal{R}}(n) \leqslant g(n)$ and $g \in \mathcal{C}$ for almost all n.*

Proof. We argue indirectly and restrict to subrecursive classes $\mathcal{C}$ that extend the class of primitive recursive functions and admit diagonalisation: there exists a function $f \colon \mathbb{N} \to \mathbb{N}$ such that for any $g \in \mathcal{C}$, g is majorised by f and $f \notin \mathcal{C}$. Furthermore we assume (without loss of generality) that three exists a recursive ordinal α_0 such that $f = \mathsf{H}_{\alpha_0}$.

By assumption for any precedence terminating TRS $\mathcal{R}$ there exists a function $g_{\mathcal{R}} \in \mathcal{C}$ such that for all terms t: $\mathsf{dheight}(t, \to_{\mathcal{R}}) \leqslant g_{\mathcal{R}}(\mathsf{sz}(t))$. Furthermore for all $g \in \mathcal{C}$ there exists $n_0 \in \mathbb{N}$ such that $g(n) < f(n)$ for all $n \geqslant n_0$.

As H_α is computable (cf. [11]), there exists a TRS $\mathcal{R}^f$ such that $\mathcal{R}^f$ is confluent, terminating and computes H_{α_0}. As $\mathcal{R}^f$ is terminating, the above proposition yields that the labelled TRS $\mathcal{R}^f_{\mathrm{lab}} \cup \mathcal{D}ec(>)$ is precedence terminating. We fix $\mathcal{R}' := \mathcal{R}^f_{\mathrm{lab}} \cup \mathcal{D}ec(>)$ and consider $g_{\mathcal{R}'} \in \mathcal{C}$. We set n such that $f(n) > g_{\mathcal{R}'}(n)$. Wlog, we assume that there exists a ground term s with $\mathsf{sz}(s) = n$ such that $\mathsf{dheight}(s, \to_{\mathcal{R}^f}) \geqslant f(\mathsf{sz}(s))$. We obtain a contradiction:

$$\begin{aligned} f(\mathsf{sz}(s)) > g_{\mathcal{R}'}(\mathsf{sz}(s)) &= g_{\mathcal{R}'}(\mathsf{sz}(\mathsf{lab}_\varnothing(s))) \\ &\geqslant \mathsf{dheight}(\mathsf{lab}_\varnothing(s), \to_{\mathcal{R}'}) \geqslant \mathsf{dheight}(s, \to_{\mathcal{R}^f}) \\ &\geqslant f(\mathsf{sz}(s)) \ . \end{aligned} \tag{6}$$

Here $\mathsf{lab}_\varnothing$ denotes the labelling function induced by the self-labelling of the TRS $\mathcal{R}^f$ and the empty assignment $\varnothing$. We remark that for any term t: $\mathsf{sz}(t) = \mathsf{sz}(\mathsf{lab}_\varnothing(t))$. The first inequality in (6) follows by definition of f and n. The second is a consequence of the definition of $g_{\mathcal{R}'}$ and the third inequality follows due to Corollary 2.1. $\square$

We remark that Lemma 7.2 additionally yields that no derivational complexity analysis of multiset (or lexicographic) path orders over infinite signatures are possible. For any precedence terminating TRS $\mathcal{R}$ there exists a multiset (or lexicographic) path order compatible with $\mathcal{R}$. For the next results we restrict to *finite* signatures.

Corollary 7.2. *Let $\mathcal{R}$ be a TRS, compatible with KBO, such that $\mathcal{F}$ is finite. Then* $\mathsf{dc}_{\mathcal{R}}(n) = \mathsf{Ack}(2^{O(n)}, 0)$.

Proof. As $\mathcal{F}$ is finite, $K = \max\{(\mathsf{mrk}(r) \dot{-} \mathsf{rkt}(l)) \mid l \to r \in \mathcal{R}'\}$ and $\mathsf{ar}(\mathcal{F})$ are well-defined. Theorem 7.1 yields that

$$\mathsf{dheight}(t, \to_{\mathcal{R}}) = \mathsf{Ack}(O(\max\{|\mathsf{tw}(t)|, \mathsf{max}(\mathsf{tw}(t)), K\}), 0) \ .$$

Again due to the finiteness of $\mathcal{F}$, for any $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $r_{\max}$ and $w_{\max}$ can be estimated independent of t. A similar calculation as in Corollary 7.1 thus yields:

$$\mathsf{dheight}(t, \to_{\mathcal{R}}) = \mathsf{Ack}(2^{O(\mathsf{sz}(t))}, 0) \ ,$$

from which the corollary follows. $\square$

In order to see that the bound in Corollary 7.2 is essentially optimal, one employs the following example.

Example 7.3 ([15])**.** Consider the TRS $\mathcal{R}_4$ consisting of the following rewrite rules:

$$\begin{array}{llll} 1\colon & \mathsf{i}(x) \circ (y \circ z) \to x \circ (\mathsf{i}^2(y) \circ z) & 2\colon & \mathsf{i}(x) \circ (y \circ (z \circ w)) \to x \circ (z \circ (y \circ w)) \\ 3\colon & \mathsf{i}(x) \to x & & \end{array}$$

These rules allow operations on (codes of) lists of natural numbers:

$$[] := \mathsf{e} \quad [k_0, \dots, k_n] := \mathsf{i}^{k_0}(\mathsf{e}) \circ [k_1, \dots, k_n] \ ,$$

For example, rule 2 corresponds to $[\dots, k+1, k', k'', \dots] \to [\dots, k, k'', k', \dots]$. Hofbauer showed for $t_n = [2, 0, \dots, 0]$ of length $n+2$, we obtain $\mathsf{dheight}(t_n) \geqslant \mathsf{Ack}(n, 0)$ [15]. Thus the derivational complexity of $\mathcal{R}_4$ cannot be bounded by a primitive recursive functions. We set the precedence $\succ$ and the weight function $(\mathsf{w}, 1)$ as follows:

$$\mathsf{i} \succ \circ \succ \mathsf{e} \qquad \mathsf{w}(\mathsf{i}) = \mathsf{w}(\circ) = 0 \quad \mathsf{w}(\mathsf{e}) = 1 \tag{7}$$

It is easy to check that the induced KBO $\succ_{\mathsf{kbo}}$ is compatible with $\mathcal{R}_4$.

Remark 7.1. We remark that one can even given an optimal upper bound on the derivational complexity induced by KBOs over finite signatures, see Lemma 12 and Corollary 19 in [21]. However, Lepper's proof in [21] cannot be extended to infinite signatures.

The next result follows directly from the definitions.

Corollary 7.3. *Let $\mathcal{R}$ be a TRS, compatible with KBO, such that $\mathcal{F}$ is finite. Then* $\mathsf{rc}_{\mathcal{R}}(n) = \mathsf{Ack}(2^{O(n)}, 0)$.

It remains to show that upper bound on the runtime complexity expressed in Corollary 7.3 is (essentially) optimal.

Example 7.4 (Example 7.3 continued)**.** We extend the TRS $\mathcal{R}_4$ from Example 7.3 by the following rules and denote the resulting TRS as $\mathcal{R}_5$:

$$4\colon \mathsf{h}(\mathsf{0}) \to \mathsf{e} \circ \mathsf{e} \qquad 5\colon \mathsf{h}(\mathsf{s}(n)) \to \mathsf{e} \circ \mathsf{h}(n)$$
$$6\colon \mathsf{g}(n) \to \mathsf{i}^2(\mathsf{e}) \circ \mathsf{h}(n) \ .$$

Then $\mathsf{g}(\mathsf{s}^n(\mathsf{0})) \xrightarrow{*}_{\mathcal{R}_5} t_n$ using the new rules and we set $r_n := \mathsf{g}(\mathsf{s}^n(\mathsf{0}))$. We conclude $\mathsf{dheight}(r_n) \geqslant \mathsf{Ack}(n, 0)$. As r_n is a basic term and $\mathsf{sz}(r_n) = O(n)$, we conclude $\mathsf{rc}_{\mathcal{R}_5}(n) \geqslant \mathsf{Ack}(n, 0)$. It remains to verify that $\mathcal{R}_5$ is compatible with KBO. For that it suffices to extend the precedence and weight function defined in (7) to fulfil the following constraints: $\mathsf{h} \succ \circ$ and $\mathsf{h} \succ \mathsf{e}$ in an admissible way and set $\mathsf{w}(\mathsf{h}) = \mathsf{w}(\mathsf{0}) = 1$ and $\mathsf{w}(\mathsf{g}) = 3$.

8 Application

In this section we consider applications of Corollary 7.1 to termination proofs of rewrite systems that employ semantic labelling. In this way we see that Corollary 7.1 is applicable to non simply terminating TRSs.

Example 8.1 (Example 3.1 continued)**.** Consider the TRS $\mathcal{R}_6$ consisting of the following rewrite rules:

$$1\colon \mathsf{f}(\mathsf{h}(x)) \to \mathsf{f}(\mathsf{j}(x)) \qquad 2\colon \mathsf{h}(\mathsf{a}) \to \mathsf{b}$$
$$3\colon \mathsf{g}(\mathsf{j}(x)) \to \mathsf{g}(\mathsf{h}(x)) \qquad 4\colon \mathsf{j}(\mathsf{a}) \to \mathsf{b} \ .$$

It is not difficult to see that termination of $\mathcal{R}$ cannot be established directly with KBO (or any other path order for that matter). However termination can be established via semantic labelling.

We use natural numbers as semantics and label only the function symbol f: $L_{\mathsf{f}} := \mathbb{N}$ and for all $g \in \mathcal{F} \setminus \{\mathsf{f}\}$, $L_g := \varnothing$. For f we employ the labelling function $\ell_{\mathsf{f}}(n) = n$. Let $\mathcal{F}_{\text{lab}}$ denote the labelled signature. As interpretation for the function symbols in $\mathcal{F}_{\text{lab}}$ we use:

$$\mathsf{a}_{\mathcal{N}} = \mathsf{b}_{\mathcal{N}} = \mathsf{g}_{\mathcal{N}}(n) = \mathsf{f}_{\mathcal{N}}(n) = 1 \qquad \mathsf{j}_{\mathcal{N}}(n) = n \qquad \mathsf{h}_{\mathcal{N}}(n) = n + 1 \ .$$

The resulting algebra $(\mathcal{N}, >)$ (with domain $\mathbb{N}$) is a quasi-model for $\mathcal{R}_6$ and the resulting labelled TRS is $\mathcal{R}_2$ (see Example 3.1). Thus, for any term $s \in \mathcal{T}(\mathcal{F}_{\text{lab}}, \mathcal{V})$ with $\mathsf{sz}(s) = n$ we obtain $\mathsf{dheight}(s, \to_{\mathcal{R}_2}) = \mathsf{Ack}(2^{O(n)}, 0)$, cf. Example 7.1. In order to exploit this to bound the derivation height of $\mathcal{R}_6$, we employ Corollary 2.1 to observe that for all

$t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$: $\mathsf{dheight}(t, \to_{\mathcal{R}_6}) \leqslant \mathsf{dheight}(\mathsf{lab}_\alpha(t), \to_{\mathcal{R}_2})$ for arbitrary assignments α. As $\mathsf{sz}(t) = \mathsf{sz}(\mathsf{lab}_\alpha(t))$ the above calculation yields

$$\mathsf{dheight}(t, \to_{\mathcal{R}_6}) \leqslant \mathsf{dheight}(\mathsf{lab}_\alpha(t), \to_{\mathcal{R}_2}) \leqslant \mathsf{Ack}(c^n, 0) \ .$$

Note that c depends only on $\mathcal{F}_{\text{lab}}$, $\mathcal{R}_2$ and on $\succ_{\mathsf{kbo}}$ employed in Example 3.1.

We emphasise that practically the upper bound on the derivation height for $\mathcal{R}_6$ obtained in Example 8.1 is not impressive. Our tool $\mathsf{T_CT}$ can verify *linear* derivational complexity in less than a second [3]. The interest lies in the fact that semantic labelling based on infinite signatures can be applied for complexity analysis. In the remainder of this section, we stress that the method is also applicable to obtain bounds on the derivational height of non simply terminating TRSs, a feature shared with matrix interpretations [32] (see [27] for recent work on matrix interpretations and complexity).

Example 8.2 (Example 7.3 continued)**.** Consider the TRS $\mathcal{R}_7$, which extends the TRS $\mathcal{R}_4$ from Example 7.3 by the following rule:

$$4\colon \mathsf{a}(\mathsf{a}(x)) \to \mathsf{a}(\mathsf{b}(\mathsf{a}(x))) \ .$$

Due to the rule 1–3, the derivational complexity of $\mathcal{R}$ cannot be bounded by a primitive recursive function. Furthermore, due to rule 4, $\mathcal{R}_7$ is not simply terminating.

Termination of $\mathcal{R}_7$ can be shown by semantic labelling, where the natural numbers are used as semantics and as labels. The following interpretations give rise to a quasi-model:

$$\mathsf{a}_{\mathcal{N}}(n) = n + 1 \qquad \mathsf{b}_{\mathcal{N}}(n) = \max(\{0, n - 1\}) \qquad \mathsf{i}_{\mathcal{N}}(n) = n \qquad m \circ_{\mathcal{N}} n = m + n \ .$$

Using the labelling function $\ell_a(n) = n$, termination of the labelled TRS in conjunction with the suitable defined TRS $\mathcal{D}ec$ (denoted as $\mathcal{R}_8$) can be shown by an instance $\succ_{\mathsf{kbo}}$ of KBO with weight function $(\mathsf{w}, 1)$: $\mathsf{w}(\circ) = \mathsf{w}(\mathsf{i}) = 0$, $\mathsf{w}(\mathsf{b}) = 1$, and $\mathsf{w}(\mathsf{a}_n) = n$. As precedence we use:

$$\mathsf{i} \succ \circ \succ \ldots \mathsf{a}_{n+1} \succ \mathsf{a}_n \succ \cdots \succ \mathsf{a}_0 \succ \mathsf{b} \ .$$

Clearly the arities of the symbols in the labelled signature are bounded. Further, we see that following definition of M is well-defined:

$$M = \{\mathsf{sp}(r) \mid l \to r \in \mathcal{R}_8\} \cup \{(\mathsf{mrk}(r) \dot{-} \mathsf{rkt}(l)) \mid l \to r \in \mathcal{R}_8\} \ .$$

Proceeding as in Example 8.1, we see that for each $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, there exists a constant c (depending on t, $\mathcal{R}'$ and $\succ_{\mathsf{kbo}}$) such that $\mathsf{dheight}(t, \to_{\mathcal{R}}) \leqslant \mathsf{Ack}(c^n, 0)$.

9 Generalised Knuth-Bendix Orders

A *weakly monotone* $\mathcal{F}$-algebra $(\mathcal{A}, \sqsupset)$ is an $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\colon f \in \mathcal{F}\})$ together with a proper order $\sqsupset$ on A such that for each function symbol f, $f_{\mathcal{A}}$ is weakly monotone; $(\mathcal{A}, \sqsupset)$ fulfils the *subterm property* if for all n-ary function symbols f, $f_{\mathcal{A}}(a_1, \ldots, a_n) \sqsupset a_i$, where $i \in \{1,\ldots,n\}$ and $a_j \in A$ for all $j = 1,\ldots,n$. If $(\mathcal{A}, \sqsupset)$ fulfils the subterm property we call it *simple*. We write $\sqsupseteq$ for the reflexive closure of $\sqsupset$ and denote with $\sqsupset_{\mathcal{A}}$ ($\sqsupseteq_{\mathcal{A}}$) the usual extension of the orders on A to orders on terms, see Section 2.

The next definition introduces *generalised Knuth-Bendix Orders* (*GKBO* for short) as proposed by Middeldorp and Zantema. Our definition is a slight restriction where all function symbols have lexicographic status and arguments are compared from left to right.

Definition 9.1. Let $\succ$ denote a precedence on $\mathcal{F}$ and let $(\mathcal{A}, \sqsupset)$ denote a weakly monotone and simple $\mathcal{F}$-algebra. Then $s \succ_{\mathsf{gkbo}} t$ holds if

1. $s \sqsupset_{\mathcal{A}} t$, or
2. $s \sqsupseteq_{\mathcal{A}} t$ and one of the following alternatives holds:
   a) $s = f(s_1,\ldots,s_n)$, $t = g(t_1,\ldots,t_m)$, and $f \succ g$.,
   b) $s = f(s_1,\ldots,s_n)$, $t = f(t_1,\ldots,t_n)$, and there exists $i \in \{1,\ldots,n\}$ such that $s_1 = t_1, \ldots, s_{i-1} = t_{i-1}$ and $s_i \succ_{\mathsf{gkbo}} t_i$.

Proposition 9.1 ([25])**.** *Let $\succ$ be a well-founded precedence and let $(\mathcal{A}, \sqsupset)$ be a weakly monotone $\mathcal{F}$-algebra such that $\sqsupset$ is well-founded, then the induced GKBO $\succ_{\mathsf{gkbo}}$ is a reduction order.*

We restrict the order type of the precedence $\succ$ and the order type $\sqsupset$ on $\mathcal{A}$ to ω. As before we assume the signature only admits bounded arities. This implies that the construction in Section 4 is applicable and we obtain the following result.

Lemma 9.1. *Let $\succ$ be a precedence with $\mathsf{otype}(\succ) = \omega$ and let $(\mathcal{A}, \sqsupset)$ be a weakly monotone $\mathcal{F}$-algebra such that $\mathsf{otype}(\sqsupset) = \omega$. Then $\mathsf{otype}(\succ_{\mathsf{gkbo}}) \leqslant \omega^\omega$.*

Proof. As $\mathsf{otype}(\sqsupset) \leqslant \omega$ we can assume without loss of generality that for the domain A of $\mathcal{A}$ we have $A \subseteq \mathbb{N}$. We adapt the earlier embedding $\mathsf{tw}\colon \mathcal{T}(\mathcal{F}, \mathcal{V}) \to \mathbb{N}^*$ and overload the notation. Let $m_0 \in A$ be minimal with respect to $\sqsupset$ and let $\alpha_0\colon \mathcal{V} \to A$ be defined as: $\alpha_0(x) := a_0$ for all $x \in \mathcal{V}$. Furthermore, set $b := \max\{\mathsf{ar}(\mathcal{F}), 2\} + 1$.

$$\mathsf{tw}(t) := \begin{cases} (m_0, 0) \frown 0^\ell & \text{if } t \text{ is a variable },\\ ([t]_{\mathcal{A}}, \mathsf{rk}(g)) \frown \mathsf{tw}(t_1) \frown \cdots \frown \mathsf{tw}(t_n) \frown 0^\ell & \text{if } t = (t_1,\ldots,t_n)\ . \end{cases}$$

The number ℓ is set suitably, so that $|\mathsf{tw}(t)| = b^{[t]_{\mathcal{A}}+1}$. Here $[\cdot]_{\mathcal{A}}$ denotes the evaluation function of $\mathcal{A}$ fixed to the assignment α_0. The proof of the lemma follows the proof of Lemma 4.1: In order to show well-definedness we argue by induction on t and exploit the fact the algebra $\mathcal{A}$ is simple. Finally the embedding is shown by induction on $s \succ_{\mathsf{gkbo}} t$. We omit the details.

□

For the next lemma, we restrict to well-founded and finitely branching relations on the algebra $(\mathcal{A}, \sqsupset)$. For applications of GKBOs, where the algebra is defined via interpretations into numbers—as exemplified below—this restriction is fulfilled.

Lemma 9.2. *Let $\mathcal{F}$ be finite and $\succ_{\mathsf{gkbo}}$ be a GKBO induced by a precedence $\succ$ of order type ω and a weakly-monotone simple algebra $(\mathcal{A}, \sqsupset)$ such that $\sqsupset$ is well-founded and finitely branching. Then* $\mathsf{otype}(\succ_{\mathsf{gkbo}}) = \omega$.

Proof. We prove that the following set is finite:

$$T := \{t \in \mathcal{T}(\mathcal{F}) \mid [t]_{\mathcal{A}} = a\} ,$$

where $a \in \mathcal{A}$. Suppose T is finite, then the number of terms with equal value is finite and thus the order type of $\succ_{\mathsf{gkbo}}$ on $\mathcal{T}(\mathcal{F})$ equals ω. As we assume $\mathcal{F}$ contains at least a constant we can embed $\succ_{\mathsf{gkbo}}$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ into $\succ_{\mathsf{gkbo}}$ on $\mathcal{T}(\mathcal{F})$. The result follows.

In order to show finiteness of T, let $k = \mathsf{dheight}(a, \sqsupset)$ and we consider the set $T' := \{t \in \mathcal{T}(\mathcal{F}) \mid \mathsf{dp}(t) \leqslant k\}$. We claim $T \subseteq T'$. Suppose otherwise $t \in T$ and $\mathsf{dp}(t) > k$. Then there exists a sequence $t_1, t_1, \ldots, t_{k+1}$ of at least $k+1$ proper subterms of t such that $t \rhd t_1 \rhd t_2 \rhd \cdots \rhd t_{k+1}$ holds. As $\sqsupset$ admits the subterm property this implies the existence of the following decreasing sequence elements of A:

$$a = [t]_{\mathcal{A}} \sqsupset [t_1]_{\mathcal{A}} \sqsupset \cdots \sqsupset [t_{k+1}]_{\mathcal{A}} ,$$

which implies that $\mathsf{dheight}(a, \sqsupset) > k$, contrary to our assumption. $\square$

The arguments given in Sections 5–7 generalise without effort to GKBOs and we obtain the following variant of Theorem 7.1.

Theorem 9.1. *Let $\mathcal{R}$ be a TRS over a signature $\mathcal{F}$ of bounded arity. Suppose $\mathcal{R}$ is compatible with GKBO $\succ_{\mathsf{gkbo}}$ induced by precedence $\succ$ and a weakly-monotone simple algebra $(\mathcal{A}, \sqsupset)$. We request $\mathsf{otype}(\succ) = \mathsf{otype}(\sqsupset) = \omega$ and assume the set $M := \{(\mathsf{mrk}(r) \dot{-} \mathsf{rkt}(l)) \mid l \to r \in \mathcal{R}\}$ is finite and set $K := \max M$. Then we have* $\mathsf{dheight}(r, \to_{\mathcal{R}}) = \mathsf{Ack}(O(\max\{|\mathsf{tw}(t)|, \max(\mathsf{tw}(t)), K\}), 0)$.

Proof. We adapt the auxiliary functions rkt and mrk defined in Section 5 to terms over $\mathcal{F}$ and employ the measure max to sequence of natural numbers obtained by the embedding tw as defined in Lemma 9.1. Then we proceed as in the proof of Theorem 7.1. $\square$

For fixed $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ and fixed algebra $(\mathcal{A}, \sqsupset)$ we can bound the argument of the Ackermann function in the above theorem in terms of the size of t. We exemplify this for algebras $(\mathcal{A}, >)$ with domain $\mathbb{N}$, where all function symbols are interpreted as strongly linear polynomials. Let $f \in \mathcal{F}$ be n-ary, then $f_{\mathcal{A}}(m_1, \ldots, m_n) = \sum_{i=1}^{n} m_i + k$ for $k \in \mathbb{N}$. Clearly such strongly linear interpretations are just weight functions and such GKBOs are equivalent to KBOs without special symbol. Let $m_0 \in \mathbb{N}$ denote the smallest number in the domain of $\mathcal{A}$ and we write $\mathsf{w}(f)$ for the weight given to f in $\mathcal{A}$. We define

$$r_{\max} := \mathsf{mrk}(t) \qquad\qquad w_{\max} := \max(\{\mathsf{w}(f) \mid f \in \mathcal{F}(t)\} \cup \{m_0\}) ,$$

Corollary 9.1. *Let $\mathcal{R}$ be a TRS that fulfils the properties in the theorem. Then there exists a constant c—depending on $\mathcal{R}$, $r_{\max}$, and $w_{\max}$ only—such that* $\mathsf{dheight}_{\mathcal{R}}(t) \leqslant \mathsf{Ack}(c^n, 0)$, *whenever* $\mathsf{sz}(t) \leqslant n$. *More succinctly we have* $\mathsf{dheight}(t, \to_{\mathcal{R}}) = \mathsf{Ack}(2^{O(\mathsf{sz}(t))}, 0)$.

We remark that for finite signatures Corollary 9.1 yields a gross overestimation of the derivation height. As GKBOs over strongly linear interpretations are equivalent to KBO without special symbol, we conclude that $\mathsf{dc}_{\mathcal{R}}(n) = 2^{O(n)}$, cf. [15, Proposition 5.14]. In what follows, we assume that $\mathcal{F}$ is a *finite signature.*

Generalised Knuth-Bendix orders (over finite signatures) characterise simple termination. For a given simply terminating TRS $\mathcal{R}$ consider the term algebra $\mathcal{T}$ where every term is interpreted by itself and order $\mathcal{T}$ by the rewrite relation $\xrightarrow{+}_{\mathcal{R}}$. It is easy to see that $(\mathcal{T}, \xrightarrow{+}_{\mathcal{R}})$ is weakly-monotone (even monotone) and admits the subterm property (as $\mathcal{R}$ is simply terminating). We fix an arbitrary precedence on $\mathcal{F}$. The induced GKBO $\succ_{\mathsf{gkbo}}$ is compatible with $\mathcal{R}$. Furthermore any TRS compatible with a GKBO $\succ_{\mathsf{gkbo}}$ is simply terminating as $\succ_{\mathsf{gkbo}}$ is a simplification order. Let $\vartheta\Omega^{\omega}$ denote the *small Veblen number* employing Weiermann's notation system in [39]. We remark that the small Veblen number forms the supremum of the order type of any lexicographic path order [31].

Theorem 9.2. *Let $\mathcal{R}$ be a TRS over a finite signature and let $\succ_{\mathsf{gkbo}}$ be a GKBO over a weakly-monotone simple algebra, such that $\mathcal{R} \subseteq \succ_{\mathsf{gkbo}}$. Then there exists $\alpha < \vartheta\Omega^{\omega}$ and there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$: $\mathsf{dc}_{\mathcal{R}}(n) \leqslant \mathsf{H}_{\vartheta\Omega^{\omega}}(n)$. This bound is essentially optimal.*

Proof. By the above observation we have that compatibility of a TRS $\mathcal{R}$ with a GKBO $\succ_{\mathsf{gkbo}}$ yields that $\mathcal{R}$ is simply terminating. By Weiermann's result [39, Corollary 6.4] there exists $\alpha < \vartheta\Omega^{\omega}$ such that $\mathsf{dc}_{\mathcal{R}}$ is eventually dominated by $\mathsf{H}_{\vartheta\Omega^{\omega}}$. This establishes the upper bound.

Furthermore by Lepper's result [22, Theorem 25] for any $\alpha < \vartheta\Omega^{\omega}$ we can construct a simply terminating TRS $\mathcal{R}_{\alpha}$ whose derivational complexity eventually dominates H_{α}. As $\mathcal{R}_{\alpha}$ is simple terminating, there exists a compatible GKBO. This establishes the lower bound. $\square$

The (essentially optimal) upper bound obtained in Theorem 9.2 is incredible large and paraphrasing similar expressions in the literature we may say that generalised KBOs are really really complex.

In concluding this section we mention a remarkable connection between Lemma 9.1 and Theorem 9.2. As remarked above any simply terminating is compatible to a GKBO $\succ_{\mathsf{gkbo}}$ based on the monotone, simple and well-founded algebra $(\mathcal{T}, \xrightarrow{+}_{\mathcal{R}})$. As $\mathcal{R}$ is finite, $\xrightarrow{+}_{\mathcal{R}}$ is finitely branching. Thus Lemma 9.1 is applicable to conclude that the order type of $\succ_{\mathsf{gkbo}}$ is ω! This small order type stands in no relation to the derivational complexity induced by GKBO expressed in Theorem 9.2. In particular we conclude that Touzet's conjecture, like Cichon's conjecture fails in general.

10 Transfinite Knuth-Bendix Orders

Following Ludwig and Waldmann, we extend the above notion of weight function to ordinal weights, cf. [23] and introduce *transfinite Knuth-Bendix orders* (*TKBOs* for short).

A *weight function* is a pair (w, w_0) such that $\mathsf{w}\colon \mathcal{F} \to \omega^\omega$ and $w_0 \in \mathbb{N}$, $w_0 > 0$ such that $\mathsf{w}(c) \geqslant w_0$ if c is a constant. Let t be a term. The *weight* of t, denoted by $\mathsf{w}(t)$, is defined inductively as follows:

$$\mathsf{w}(t) := \begin{cases} w_0 & t \text{ is a variable} \\ \mathsf{w}(f) \oplus \mathsf{w}(t_1) \cdots \oplus \mathsf{w}(t_n) & \text{if } t = f(t_1, \ldots, t_n) \ . \end{cases}$$

Here $\oplus$ denotes the natural sum, cf. [17]. Admissibility of a weight function (w, w_0), where $\mathsf{w}\colon \mathcal{F} \to \omega^\omega$ is defined as before, see Section 3.

Definition 10.1. Let (w, w_0) denote an admissible weight function for a precedence $\succ$. Then $s \succ_{\mathsf{tkbo}} t$ holds if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and

1. $\mathsf{w}(s) > \mathsf{w}(t)$, or
2. $\mathsf{w}(s) = \mathsf{w}(t)$, and one of the following alternatives holds:
   a) t is a variable, $s = f^k(t)$, $k > 0$,
   b) $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$, and $f \succ g$.,
   c) $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, and there exists $i \in \{1, \ldots, n\}$ such that $s_1 = t_1, \ldots, s_{i-1} = t_{i-1}$ and $s_i \succ_{\mathsf{tkbo}} t_i$.

We recall the following fact about the TKBO from [23].

Proposition 10.1 ([23])**.** *For any precedence $\succ$, the induced TKBO $\succ_{\mathsf{tkbo}}$ is a simplification order.*

The definition of TKBOs proposed in [23] is more general in the respect that weight functions for arbitrary ordinals less than ϵ_0 are admitted and that so-called *subterm coefficient functions* are employed. The latter generalisation is of limited interest in the present context. Furthermore in [20] Kovacs et al. we have shown that the indicated ordinal weights suffice.

We say a TKBO $\succ_{\mathsf{tkbo}}$ is *finite* if all ordinal weights are finite. In the following we re-prove the following result by Winkler et al. [40].

Proposition 10.2 ([40])**.** *If a finite TRS $\mathcal{R}$ is compatible with a TKBO, then $\mathcal{R}$ is compatible with a finite TKBO.*

Essentially the proof in [40] makes use of the observation that in a finite TRS it suffices to employ large enough natural numbers as weights and no ordinals $> \omega$ are necessary. The provided proof is combinatorial and thus needlessly restricted to specific ordinal notation systems. We generalise the argument by making use of the *slow-growing hierarchy* as collapsing functions. The family of *slow-growing functions* $(\mathsf{G}_\alpha)_{\alpha<\omega^\omega}$ is defined as follows:

$$\mathsf{G}_0(x) = 0 \qquad \mathsf{G}_{\alpha+1}(x) = \mathsf{G}_\alpha(x) + 1 \qquad \mathsf{G}_\lambda(x) = \mathsf{G}_{\lambda[x]}(x) \quad (\lambda \text{ limit}) \ .$$

The next example clarifies that the slow-growing hierarchy (over the standard assignment of fundamental sequences) is indeed slow growing.

Example 10.1.

$$\mathsf{G}_{\omega}(x) = x + 1 \qquad \mathsf{G}_{\omega^{\omega^{\omega}}}(x) = (x+1)^{x+1^{x+1}} \ .$$

We state some simple facts on the slow-growing hierarchy.

Lemma 10.1. *1. If $\alpha >_{(n)} \beta$ and $m \geqslant n$, then $\mathsf{G}_{\alpha}(m) > \mathsf{G}_{\beta}(m)$.*

2. If $n > m$ and $\alpha \geqslant \omega$, then $\mathsf{G}_{\alpha}(n) > \mathsf{G}_{\alpha}(m)$.

As a consequence of Lemma 10.1 we obtain that the family $(\mathsf{G}_{\alpha})_{\alpha \in \mathcal{O}}$ forms a hierarchy, that is, for $\alpha > \beta$ there exists c such that for all $x \geqslant c$ we have $\mathsf{G}_{\alpha}(x) > \mathsf{G}_{\beta}(x)$.

Proof of Theorem on Finite TKBOs. It suffices to show that there exists $k \in \mathbb{N}$ such that

$$\mathsf{w}(l) \geq_{(k)} \mathsf{w}(r) \qquad \text{for all rules } l \to r \in \mathcal{R} \ . \tag{8}$$

Assuming (8) we obtain $\mathsf{G}_{\mathsf{w}(l)}(n) \geqslant \mathsf{G}_{\mathsf{w}(r)}(n)$ $(n \geqslant k)$ for all rules $l \to r \in \mathcal{R}$. From this the theorem follows. In order to prove (8) we set $k := \max\{\mathsf{N}(\mathsf{w}(r)) \mid l \to r \in \mathcal{R}\}$, where the *norm* of an ordinal $\alpha < \omega^{\omega}$ is defined as follows:

$$\mathsf{N}(0) := 0 \qquad \mathsf{N}(\omega^{k_1} + \cdots \omega^{k_n}) := \sum_{i \leqslant n} k_i + n \ .$$

Exploiting the definition of fundamental sequences we see that if $\alpha > \beta$ and $n = \mathsf{N}(\beta)$, then $\alpha_n \geqslant \beta$. In particular we obtain $\alpha >_{(n)} \beta$ whenever $n \geqslant \mathsf{N}(\beta)$. From this we obtain that $\mathsf{w}(l) \geqslant \mathsf{w}(r)$ implies $\mathsf{w}(l) \geq_{(k)} \mathsf{w}(r)$ for any rule $l \to r$. □

Essentially as a consequence of Proposition 10.2 and Hofbauer's and Lepper's classifications of the derivational complexity of KBOs, we obtain the following result.

Theorem 10.1. *If $\mathcal{R}$ is a finite TRS over the signature $\mathcal{F}$ compatible with a TKBO, then $\mathsf{dc}_{\mathcal{R}}(n) = \mathsf{Ack}(O(n), 0)$ if $\mathcal{F}$ contains a special symbol and $\mathsf{dc}_{\mathcal{R}}(n) = 2^{O(n)}$ otherwise. Both bounds are optimal.*

Proof. Let $\mathcal{R}$ be finite and let $\succ_{\mathsf{tkbo}}$ be a TKBO compatible with $\succ_{\mathsf{tkbo}}$ over a precedence $\succ$ and an admissible weight function (w, w_o). Due to Proposition 10.2 there exists an equivalent KBO $\succ_{\mathsf{kbo}}$ induced by $\succ$ but an altered (but admissible) weight function (w', w_0). Without loss of generality generality, we assume $\mathcal{F}$ contains a special symbol. Then Lepper's result [21, Corollary 19] yields the existence of a $c \in \mathbb{N}$ such that $\mathsf{dc}_{\mathcal{R}}(n) \leqslant \mathsf{Ack}(c \cdot n, 0)$. Optimality follows from Example 7.3. □

We remark that the upper bounds (and their optimality) preserve when we consider runtime complexity instead of derivational complexity. It suffices to argue for optimality. For the case of TKBO over a signature containing a special symbol this directly follows from Example 7.4 and for the case without special symbol a similar adaption of Proposition 5.8 in [15] can be applied.

References

[1] T. Arts and J. Giesl. A collection of examples for termination of term rewriting using dependency pairs. Technical Report AIB-2001-09, RWTH Aachen, 2001.

[2] M. Avanzini and G. Moser. Closing the gap between runtime complexity and polytime computability. In *Proc. 21st International Conference on Rewriting Techniques and Applications*, volume 6 of *LIPIcs*, pages 33–48, 2010.

[3] M. Avanzini and G. Moser. Tyrolean complexity tool: Features and usage. In *Proc. 24th International Conference on Rewriting Techniques and Applications*, volume 21 of *LIPIcs*, pages 71–80, 2013.

[4] Martin Avanzini. *Verifying Polytime Computability Automatically.* PhD thesis, University of Innsbruck, 2013.

[5] F. Baader and T. Nipkow. *Term Rewriting and All That.* Cambridge University Press, 1998.

[6] W. Buchholz. Proof-theoretical analysis of termination proofs. *Annals of Pure and Applied Logic*, 75:57–65, 1995.

[7] C. Choppy, S. Kaplan, and M. Soria. Complexity analysis of term-rewriting systems. *Theoretical Computer Science*, 67(2–3):261–282, 1989.

[8] E.-A. Cichon. Termination orderings and complexity characterisations. In P. Aczel, H. Simmons, and S.S. Wainer, editors, *Proof Theory*, pages 171–193, 1992.

[9] E.-A. Cichon and P. Lescanne. Polynomial interpretations and the complexity of algorithms. In *Proc. 11th International Conference on Automated Deduction*, volume 607 of *LNCS*, pages 139–147, 1992.

[10] N. Dershowitz. Termination of Rewriting. *Journal of Symbolic Computation*, 3 (1–2):69–116, 1987.

[11] M. Fairtlough and S. S. Wainer. Hierarchies of provably recursive functions. In S.-R. Buss, editor, *Handbook of Proof Theory*, pages 149–207. Elsevier Science, 1998.

[12] J. Giesl, R. Thiemann, and P. Schneider-Kamp. Proving and disproving termination in the dependency pair framework. In *Deduction and Applications*, volume 05431 of *Dagstuhl Seminar Proceedings*, 2006.

[13] N. Hirokawa and A. Middeldorp. Tyrolean termination tool: Techniques and features. *Information and Computation*, 205:474–511, 2007.

[14] N. Hirokawa and G. Moser. Automated complexity analysis based on the dependency pair method. In *Proc. 4th International Joint Conference on Automated Reasoning*, volume 5195 of *LNAI*, pages 364–380, 2008.

[15] D. Hofbauer. *Termination Proofs and Derivation Lengths in Term Rewriting Systems.* PhD thesis, Technische Universität Berlin, 1991.

[16] D. Hofbauer and C. Lautemann. Termination Proofs and the Length of Derivations. In *Proc. of 3rd International Conference on Rewriting Techniques and Applications*, volume 355 of *LNCS*, pages 167–177, 1989.

[17] T. Jech. *Set Theory.* Springer Verlag, 2002.

[18] L. Kirby and J. Paris. Accessible independence results for Peano arithmetic. *Bulletin London Mathematical Society*, 4:285–293, 1982.

[19] K. Korovin and A. Voronkov. Orienting rewrite rules with the Knuth-Bendix order. *Information and Computation*, 183(2):165–186, 2003.

[20] L. Kovács, G. Moser, and A. Voronkov. On transfinite knuth-bendix orders. In *Proc. 23rd International Conference on Automated Deduction*, volume 6803 of *LNCS*, pages 384–399, 2011.

[21] I. Lepper. Derivation lengths and order types of Knuth-Bendix orders. *Theoretical Computer Science*, 269:433–450, 2001.

[22] I. Lepper. Simply terminating rewrite systems with long derivations. *Archive for Mathematical Logic*, 43:1–18, 2004.

[23] M. Ludwig and U. Waldmann. An extension of the Knuth-Bendix ordering with LPO-like properties. In *Proc. 14th International Conference on Logic for Programming Artificial Intelligence and Reasoning*, volume 4790 of *LNCS*, pages 348–362, 2007.

[24] U. Martin. How to chose weights in the Knuth-Bendix ordering. In *Proc. 2nd International Conference on Rewrite Techniques and Applications*, volume 256 of *LNCS*, pages 42–53, 1987.

[25] A. Middeldorp and H. Zantema. Simple termination of rewrite systems. *Theoretical Computer Science*, 175(1):127–158, 1997.

[26] A. Middeldorp, H. Ohsaki, and H. Zantema. Transforming termination by self-labelling. In *Proc. 13th International Conference on Automated Deduction*, volume 1104 of *LNCS*, pages 373–387, 1996.

[27] A. Middeldorp, G. Moser, F. Neurauter, J. Waldmann, and H. Zankl. Joint spectral radius theory for automated complexity analysis of rewrite systems. In *Algebraic Informatics - 4th International Conference*, volume 6742 of *LNCS*, pages 1–20, 2011.

[28] G. Moser. Derivational complexity of Knuth-Bendix orders revisited. In *Proc. 13th International Conference on Logic for Programming Artificial Intelligence and Reasoning*, volume 4246 of *LNAI*, pages 75–89, 2006.

[29] G. Moser. Proof theory at work: Complexity analysis of term rewrite systems. *CoRR*, abs/0907.5527, 2009. Habilitation Thesis.

[30] G. Moser and A. Schnabl. Termination proofs in the dependency pair framework may induce multiple recursive derivational complexity. In *Proc. 21st International Conference on Rewrite Techniques and Applications*, volume 10 of *LIPIcs*, pages 235–250, 2011.

[31] G. Moser and A. Weiermann. Relating derivation lengths with the slow-growing hierarchy directly. In *Proc. of 14th International Conference on Rewrite Techniques and Applications*, volume 2706 of *LNCS*, pages 296–310, 2003.

[32] G. Moser, A. Schnabl, and J. Waldmann. Complexity analysis of term rewriting based on matrix and context dependent interpretations. In *Proc. 28th Foundations of Software Technology and Theoretical Computer Science*, volume 2 of *LIPIcs*, pages 304–315, 2008.

[33] H.-E. Rose. *Subrecursion: Functions and Hierarchies*. Clarendon Press, 1984.

[34] A. Schnabl. *Derivational Complexity Analysis*. PhD thesis, University of Innsbruck, 2012.

[35] J. Steinbach. Extensions and comparison of simplification orderings. In *Proc. 3rd International Conference on Rewrite Techniques and Applications*, volume 355 of *LNCS*, pages 434–448, 1989.

[36] C. Sternagel and R. Thiemann. Formalizing knuth-bendix orders and knuth-bendix completion. In *Proc. 24rd International Conference on Rewrite Techniques and Applications*, volume 21 of *LIPIcs*, pages 287–302, 2013.

[37] TeReSe. *Term Rewriting Systems*, volume 55 of *Cambridge Tracks in Theoretical Computer Science*. Cambridge University Press, 2003.

[38] H. Touzet. A characterisation of multiply recursive functions with higman's lemma. *Information and Computation*, 178(2):534–544, 2002.

[39] A. Weiermann. Complexity bounds for some finite forms of Kruskal's theorem. *Journal of Symbolic Computation*, 18(5):463–488, 1994.

[40] S. Winkler, H. Zankl, and A. Middeldorp. Ordinals and knuth-bendix orders. In *Proc. 18th International Conference on Logic for Programming Artificial Intelligence and Reasoning*, volume 7180 of *LNCS*, pages 420–434, 2012.

[41] H. Zankl, N. Hirokawa, and A. Middeldorp. KBO orientability. *Journal of Automated Reasoning*, 43(2):173–201, 2009.

[42] H. Zantema. Termination of term rewriting by semantic labelling. *Fundamenta Informaticae*, 24:89–105, 1995.